Representation theory

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Contents

1. Derived algebraic geometry

In this chapter, we follow the book [GR] of Gaitsgory-Rozenblyum, and give a brief account of what derived algebraic geometry is needed in geometric representation theory.

1.1. Stacks and schemes

1.1.1. Categorical constructions

Definition 1.1.1.1. Throughout this note, the word category will mean $(\infty, 1)$-categories unless stated otherwise. A functor is said to be continuous if it preserves all filtered colimits. Most importantly, we use cohomological indexing.

remark 1.1.1.1. We assume the reader is familiar with the definition of a t-structure on stable ∞-categories but slightly confused about the indexing convention, so we briefly recall here that what in the cohomological indexing convention should the t-structure should look like. For a stable ∞-category $\mathcal{C}$, it is equipped with two subcategories $\mathcal{C}^{\leqslant 0}$ and $\mathcal{C}^{\geqslant 0}$, where the shift or suspension functor $[1] = \Sigma$ preserves $\mathcal{C}^{\leqslant 0}$ (i.e., $[1]$ shifts the grading down). Moreover, we have $\tau^{\geqslant 0} : \mathcal{C} \to \mathcal{C}^{\geqslant 0}$ left adjoint to the inclusion and $\tau^{\leqslant 0} : \mathcal{C} \to \mathcal{C}^{\leqslant 0}$ right adjoint to the inclusion, which gives rise to the semi-orthogonality $H^0\mathrm{Hom}(c, d) = 0$ for $c \in \mathcal{C}^{\leqslant 0}$ and $d \in \mathcal{C}^{\geqslant 1}$. Then, we have the decomposition for any object

$$\tau^{\leqslant 0}c \to c \to \tau^{\geqslant 1}c$$

Finally, we may define the eventually connective or coconnective parts of $\mathcal{C}$ by

$$\mathcal{C}^{-} = \bigcup_{n \geqslant 0} \mathcal{C}^{\leqslant n}, \mathcal{C}^{+} = \bigcup_{n \leqslant 0} \mathcal{C}^{\geqslant n}$$

We say a functor is left t-exact if it sends $\mathcal{C}^{\geqslant 0}$ to $\mathcal{D}^{\geqslant 0}$, and right t-exact if the dual version holds. Consequently, left t-exact functors sends $\mathcal{C}^{+}$ to $\mathcal{D}^{+}$ and right t-exact functors the dual.

Definition 1.1.1.2. We say a functor $F : \mathcal{C} \to \mathcal{D}$ is of finite tor-amplitude if it is left t-exact and right t-exact up to different shifts. We say an object $c \in \mathcal{C}$ a monoidal stable ∞-category is of finite tor-amplitude $- \otimes c : \mathcal{C} \to \mathcal{C}$ is so.

Definition 1.1.1.3. We fix a ground field k. We let $\mathbf{Vect} = \mathbf{Mod}_{Hk}(\mathbf{Sp})$ denote the category of module spectra of the Eilenberg-Maclane spectrum Hk regarded as a $\mathbb{E}_{\infty}$-ring spectrum.

Definition 1.1.1.4. We define the category $\mathbf{DGCat}$ to be $\mathbf{Mod}_{\mathbf{Vect}}(\mathbf{Pr}^{L}_{\mathrm{St}})$ (we will not go into details of the definitions and properties of $\mathbf{Pr}^{L}_{\mathrm{St}}$ here, but the readers may refer to [Lur08] for $\mathbf{Pr}^{L}$ and [Lur17] for $\mathbf{Pr}^{L}_{\mathrm{St}}$). Moreover, we can define its non-continuous version $\mathbf{DGCat}^{\mathrm{nc}}$, given as follows:

- We begin with the category $\mathbf{Mod}_{\mathbf{Vect}^{\omega}}(\mathbf{Cat})$.
- We take its full subcategory spanned by categories that are stable and having that the action functor $\mathbf{Vect}^{\omega} \times \mathcal{C} \to \mathcal{C}$ exact in both variables. We denote this category by $\mathbf{DGCat}^{\text{non-cocmpl}}$
- We define $\mathbf{DGCat}^{\mathrm{nc}}$ to be the essential image of the functor $\mathbf{DGCat} \to \mathbf{DGCat}^{\text{non-cocmpl}}$.

todo 1.1.1.1. Write something more here, at least explain left complete t-structure and blah blah.

1.1.2. The category of prestacks

Definition 1.1.2.1. We define the category of affine schemes by $\mathbf{Aff} = \mathbf{CAlg}(\mathbf{Vect}^{\leqslant 0})^{\mathrm{op}}$.

remark 1.1.2.1. The geometry of non-connective and connective schemes differ subtlely, and we (following Gaitsgory and possibly Lurie) consider the connective case in most circumstances.

Definition 1.1.2.2. We define the category of prestacks by $\mathbf{pStk} = \mathrm{pSh}(\mathbf{Aff}) = \mathrm{Fun}(\mathbf{Aff}^{\mathrm{op}}, \mathbf{Ani})$. By Yoneda embedding, we obtain an inclusion $\mathbf{Aff} \hookrightarrow \mathbf{pStk}$.

Definition 1.1.2.3. We now introduce a truncated version of affine schemes. Consider the subcategory of $\mathbf{Vect}$ given by the objects concentrated between degrees $[-n, 0]$, say $\mathbf{Vect}^{[-n,0]}$. We have a symmetric monoidal structure on this subcategory given by $(A, B) \mapsto \tau^{\geqslant -n}(A \otimes B)$, thus we may define the category of n-coconnective affine schemes ${}^{\leqslant n}\mathbf{Aff} = \mathbf{CAlg}\big(\mathbf{Vect}^{[-n,0]}\big)^{\mathrm{op}}$. This category of commutative algebras is denoted by $\mathbf{CAlg}^{[-n,0]}$ also for short.

Construction 1.1.2.4. Notice that the inclusion $\mathbf{Vect}^{[-n,0]} \hookrightarrow \mathbf{Vect}^{\leqslant 0}$ is lax symmetric monoidal, and inparticular preserves commutative algebras. Thus we have a fully faithful functor

$$\mathbf{CAlg}^{[-n,0]} \hookrightarrow \mathbf{CAlg}^{\leqslant 0}$$

hence an inclusion ${}^{\leqslant n}\mathbf{Aff} \hookrightarrow \mathbf{Aff}$.

Notice that we have a truncation functor $\tau^{\geqslant -n} : \mathbf{Vect}^{\leqslant 0} \to \mathbf{Vect}^{[-n,0]}$, which naturally restricts to the commutative algebra category, that is, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{CAlg}^{\leqslant 0} \arrow[r, "\tau^{\geqslant -n}"] \arrow[d, "U"'] & \mathbf{CAlg}^{[-n,0]} \arrow[d, "U"] \\
\mathbf{Vect}^{\leqslant 0} \arrow[r, "\tau^{\geqslant -n}"'] & \mathbf{Vect}^{[-n,0]}
\end{tikzcd}$$

where U is the forgetful functor. Moreover, $\tau^{\geqslant -n}$ is left adjoint to the inclusion functor. We denote the functor induced on the category of affine schemes by taking op by ${}^{\leqslant n}\tau : \mathbf{Aff} \to {}^{\leqslant n}\mathbf{Aff}$. We abuse notations and write the same functor post-composed with the inclusion ${}^{\leqslant n}\mathbf{Aff} \hookrightarrow \mathbf{Aff}$.

Definition 1.1.2.5. We have ${}^{\leqslant 0}\mathbf{Aff}$ by definition the classical category of affine schemes. Hence, we will also denote ${}^{\leqslant 0}\tau(-)$ by $(-)_{\mathrm{cl}}$

Definition 1.1.2.6. We define the category

$$^{<\infty}\mathbf{Aff} = \bigcup {}^{\leqslant n}\mathbf{Aff}$$

the category of eventually coconnective affine schemes.

Definition 1.1.2.7. We define the category of n-prestacks by $^{\leqslant n}\mathbf{pStk} = \mathrm{pSh}(^{\leqslant n}\mathbf{Aff}^{\mathrm{op}})$. Restriction along the inclusion $^{\leqslant n}\mathbf{Aff} \hookrightarrow \mathbf{Aff}$ gives a functor $^{\leqslant n}\tau : \mathbf{pStk} \to {}^{\leqslant n}\mathbf{pStk}$. Moreover, the left Kan extension functor gives a fully faithful left adjoint of $^{\leqslant n}\tau$, say $\mathrm{LKE} : {}^{\leqslant n}\mathbf{pStk} \hookrightarrow \mathbf{pStk}$, which somehow justifies the name of the above truncation functor.

remark 1.1.2.2. Right Kan extension gives a right adjoint of the functor $^{\leqslant n}\tau$.

Warning 1.1.2.8. We should think $^{\leqslant n}\mathbf{pStk}$ as the prestacks n-coconnective in the sense that they are detected or determined by all the maps from n-coconnective affines, rather than the usual sense of truncatedness of presheaves, i.e., has values in n-coconnective vector spaces.

Definition 1.1.2.9. Let $\mathcal{X}$ be a prestack. We say $\mathcal{X}$ is convergent (or nilcomplete in the sense of [Lur18]) if for any $S \in \mathbf{Aff}$, we have $\mathcal{X}(S) \to \lim_n \mathcal{X}(^{\leqslant n}\tau(S))$ is an equivalence.

Lemma 1.1.2.10. Any prestack representable by an affine scheme is convergent.

Proof: By completeness of the t-structure in $\mathbf{Sp}$ hence in $\mathbf{Vect}$, for any algebra $A \in \mathbf{CAlg}(\mathbf{Vect}^{\leqslant 0})$ we have $A \to \lim \tau^{\leqslant n} A$ is an isomorphism. The rest follows by Yoneda embedding and the universal property of $\tau^{\leqslant n}$ as a left adjoint. □

Example 1.1.2.3. The stack of isomorphisms of quasi-coherent sheaves is not convergent.

Proposition 1.1.2.11. A prestack $\mathcal{X}$ is convergent iff as a functor $\mathcal{X} : (\mathbf{Aff})^{\mathrm{op}} \to \mathbf{Ani}$, we have the equivalence $\mathcal{X} \simeq \mathrm{RKE}\big(\mathcal{X}|_{(^{<\infty}\mathbf{Aff})^{\mathrm{op}}}\big)$

Proof: We first directly compute the right Kan extension of $\mathcal{Z} : (^{<\infty}\mathbf{Aff})^{\mathrm{op}} \to \mathbf{Ani}$. By the point-wise limit formula,

$$\mathcal{Z}(S) = \lim_{S' \in (^{<\infty}\mathbf{Aff})_{/S}} \mathcal{Z}(S')$$

So it suffices to see that the functor $\mathbb{N} \to (^{<\infty}\mathbf{Aff})_{/S}$ by sending n to $^{\leqslant n}\tau(S)$ is cofinal in the category. We omit the higher bullshit here and verify only that it is cofinal on the level of underlying 1-categories. By definition, any object $T \in {}^{<\infty}\mathbf{Aff}$ is n-truncated for some n, and a map $T \to S$ hence factors through $T \to {}^{\leqslant n}\tau(S)$ by adjunction. □

Construction 1.1.2.12. By the above proposition, if we let $\mathbf{pStk}_{\mathrm{conv}} \subset \mathbf{pStk}$ denote the subcategory of convergent prestacks, this inclusion admit a left adjoint $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{conv}}$, constructed as $\mathcal{X}_{\mathrm{conv}} = \mathrm{RKE}(\mathcal{X}|_{<\infty \mathbf{Aff}})$

Beside coconnectivity, there is a parallel notion we can consider on stacks. We denote this property by truncatedness.

Definition 1.1.2.13. We say a prestack $\mathcal{X}$ is k-truncated if as a functor $\mathbf{Aff}^{\mathrm{op}} \to \mathbf{Ani}$, it has image in $\mathbf{Ani}_{\leqslant k}$. We denote this category by $\mathbf{pStk}_{\leqslant k}$. Likewise, we may define the category ${}^{\leqslant n}\mathbf{pStk}_{\leqslant k}$. This form a $(k, 1)$-category.

Example 1.1.2.4. By definition, the Yoneda image of ${}^{\leqslant n}\mathbf{Aff}^{\mathrm{op}}$ in ${}^{\leqslant n}\mathbf{pStk}$ is n-truncated.

1.1.3. Finite type conditions

Definition 1.1.3.1. We say an object $A \in \mathbf{CAlg}^{\leqslant 0}$ is of finite type if $H^0(A)$ is of finite type over k and $H^i(A)$ is finitely generated module over $H^0(A)$. We denote this category by $\mathbf{CAlg}^{\leqslant 0}_{\mathrm{ft}}$. We denote the intersection $\mathbf{CAlg}^{\leqslant 0}_{\mathrm{ft}} \cap \mathbf{CAlg}^{[-n,0]} = \mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}}$. Likewise on its opposite, we define ${}^{<\infty}\mathbf{Aff}_{\mathrm{ft}}$ and ${}^{<\infty}\mathbf{Aff}_{\mathrm{ft}} \cap {}^{\leqslant n}\mathbf{Aff} = {}^{\leqslant n}\mathbf{Aff}_{\mathrm{ft}}$.

Theorem 1.1.3.2 ([Lur17]). The objects of $\mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}}$ are compact in $\mathbf{CAlg}^{[-n,0]}$. Moreover, for any $S \in {}^{\leqslant n}\mathbf{Aff}$, the category opposite to $I = \left({}^{\leqslant n}\mathbf{Aff}_{\mathrm{ft}}\right)_{/S}$ is filtered, and the cofiltered limit $S \mapsto \lim_{S_0 \in I} S_0$ is an isomorphism.

Corollary 1.1.3.3. $\mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}}$ is a subcategory of compact generators in $\mathbf{CAlg}^{[-n,0]}$. Hence, $\mathbf{CAlg}^{[-n,0]} = \mathrm{Ind}\left(\mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}}\right)$. Dually, we have ${}^{\leqslant n}\mathbf{Aff} = \mathrm{Pro}({}^{\leqslant n}\mathbf{Aff}_{\mathrm{ft}})$. Moreover, since $\mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}}$ is idempotent complete, we have $\mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}} = \left(\mathbf{CAlg}^{[-n,0]}\right)^{\omega}$.

Definition 1.1.3.4. Let $\mathcal{X} \in {}^{\leqslant n}\mathbf{pStk}$ for some n, we say it is locally of finite type if it is the left Kan extension $\mathcal{X} = \mathrm{LKE}\left(\mathcal{X}|_{{}^{\leqslant n}\mathbf{Aff}_{\mathrm{ft}}}\right)$. We denote the full subcategory by ${}^{\leqslant n}\mathbf{pStk}_{\mathrm{lft}}$.

Construction 1.1.3.5. Regarding the above definition as ${}^{\leqslant n}\mathbf{pStk}_{\mathrm{lft}} = \mathrm{pSh}({}^{\leqslant n}\mathbf{Aff}_{\mathrm{ft}})$, we may restate the above definition as there is an inclusion functor given by the left Kan extension ${}^{\leqslant n}\mathbf{pStk}_{\mathrm{lft}} \hookrightarrow {}^{\leqslant n}\mathbf{pStk}$ with a right adjoint given by restriction of sites.

Corollary 1.1.3.6. By Theorem 1.1.3.2, a stack $\mathcal{X} \in {}^{\leqslant n}\mathbf{pStk}$, equivalent to a functor $\mathbf{CAlg}^{[-n,0]} \to \mathbf{Ani}$ is left Kan extended from $\mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}} = \left(\mathbf{CAlg}^{[-n,0]}\right)^{\omega}$ iff it preserves filtered colimits. Thus, by expanding definitions, ${}^{\leqslant n}\mathbf{Aff} \cap {}^{\leqslant n}\mathbf{pStk}_{\mathrm{lft}} = {}^{\leqslant n}\mathbf{Aff}_{\mathrm{ft}}$.

Definition 1.1.3.7. We say an affine scheme $S \in \mathbf{Aff}$ is almost of finite type if for all n we have ${}^{\leqslant n}\tau(S)$ is of finite type. We denote the full subcategory of almost of finite type affine schemes by ${}^{\leqslant n}\mathbf{Aff}_{\mathrm{aft}}$.

Dually, an algebra $A \in \mathbf{CAlg}^{\leqslant 0}$ is almost of finite type if $H^0(A)$ is of finite type over k and $H^n(A)$ is finitely generated as an $H^0(A)$-module (for example, this implies that A is coherent in the sense of [Lur17], Definition 7.2.4.16, since we are working over fields). We denote the category by $\mathbf{CAlg}^{\leqslant 0}_{\mathrm{aft}}$.

Definition 1.1.3.8. We say a prestack $\mathcal{X} \in \mathbf{pStk}$ is locally almost of finite type if it is convergent and for every n, we have ${}^{\leqslant n}\tau(\mathcal{X}) \in {}^{\leqslant n}\mathbf{pStk}_{\mathrm{lft}}$. We denote the corresponding full subcategory by $\mathbf{pStk}_{\mathrm{laft}}$.

Corollary 1.1.3.9. By Corollary 1.1.3.6, $\mathbf{pStk}_{\mathrm{laft}} \cap \mathbf{Aff} = \mathbf{Aff}_{\mathrm{aft}}$.

Proposition 1.1.3.10. We have an equivalence of categories

$$\mathbf{pStk}_{\mathrm{laft}} = \mathrm{pSh}({}^{<\infty}\mathbf{Aff}_{\mathrm{ft}})$$

where $\mathbf{pStk}_{\mathrm{laft}} \to \mathrm{pSh}({}^{<\infty}\mathbf{Aff}_{\mathrm{ft}})$ is given by restricting along the inclusion ${}^{<\infty}\mathbf{Aff}_{\mathrm{ft}} \hookrightarrow \mathbf{Aff}$, and the inverse is given by $\mathrm{RKE}_{({}^{<\infty}\mathbf{Aff} \hookrightarrow \mathbf{Aff})} \mathrm{LKE}_{({}^{<\infty}\mathbf{Aff}_{\mathrm{aft}} \hookrightarrow {}^{<\infty}\mathbf{Aff})} : \mathrm{pSh}({}^{<\infty}\mathbf{Aff}_{\mathrm{aft}}) \to \mathrm{pSh}({}^{<\infty}\mathbf{Aff}) \to \mathrm{pSh}(\mathbf{Aff})$.

Proof: Recall that we require a locally almost of finite type prestack $\mathcal{X}$ to be convergent. By Proposition 1.1.2.11, $\mathcal{X}$ is determined by its data on ${}^{<\infty}\mathbf{Aff}^{\mathrm{op}}$. It suffices to prove that the data of $\mathcal{X}$ on ${}^{<\infty}\mathbf{Aff}^{\mathrm{op}}$ is determined by its data on ${}^{<\infty}\mathbf{Aff}^{\mathrm{op}}_{\mathrm{aft}}$ via left Kan extension. By definition, the restriction ${}^{\leqslant n}\tau(\mathcal{X}) : {}^{\leqslant n}\mathbf{Aff}^{\mathrm{op}} \to \mathbf{Ani}$ is obtained from left Kan extension from ${}^{\leqslant n}\mathbf{Aff}^{\mathrm{op}}_{\mathrm{ft}}$. Thus, it suffices to identify the following two conditions on a convergent stack $\mathcal{X}$ regarded as a functor $\mathcal{X} : \mathbf{CAlg}^{\leqslant 0} \to \mathbf{Ani}$:

- It is a Kan extension along $\mathbf{CAlg}^{\leqslant 0}_{\mathrm{aft}} \hookrightarrow \mathbf{CAlg}^{\leqslant 0}$.
- Its restriction to any $\mathbf{CAlg}^{[-n,0]}$ is the Kan extension along $\mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}} \hookrightarrow \mathbf{CAlg}^{[-n,0]}$.

Notice that by definition $\mathbf{CAlg}^{[-n,0]} \cap \mathbf{CAlg}^{\leqslant 0}_{\mathrm{aft}} = \mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}}$, then note for $A \in \mathbf{CAlg}^{[-n,0]}$, $B \in \mathbf{CAlg}^{\leqslant 0}$, we can compute $H^0\mathrm{Hom}(B, A) = H^0\mathrm{Hom}(\tau^{\geqslant -n}B, A)$.

We first show the first condition implies the second. In the pointwise colimit formula for computing the value at A, namely

$$\mathrm{LKE}(F)(A) = \operatorname*{colim}_{B \to A} F(B)$$

we may assume $B \in \mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}}$ since any map $B \to A$ factors through $B \to \tau^{\geqslant -n}B \to A$, exhibiting the subcategory $\left(\mathbf{CAlg}^{[-n,0]}_{\mathrm{ft}}\right)_{/A}$ cofinal in $\left(\mathbf{CAlg}^{\leqslant 0}_{\mathrm{aft}}\right)_{/A}$. Thus, the first condition clearly implies the second.

The converse is done similarly. □

The above arguments give a basic setting for performing descent to extend convergent functors from affine schemes to prestacks.

1.1.4. Descent conditions and stacks

Definition 1.1.4.1. Recall a morphism $A \to B$ of $\mathbb{E}_\infty$-rings in $\mathbf{Vect}^{\leqslant 0}$ is flat if $H^0(A) \to H^0(B)$ is flat, and the map $H^n(A) \otimes_{H^0(A)} H^0(B) \to H^n(B)$ is an isomorphism. For $S, S' \in \mathbf{Aff}$, we say a morphism $S \to S'$ is flat if it is so on the structure rings.

By definition, the notion of flatness is convergent, i.e., a map $S \to S'$ is flat iff ${}^{\leqslant n}\tau(S) \to {}^{\leqslant n}\tau(S')$ is so.

Definition 1.1.4.2. We say a morphism of affine schemes $f : S \to S'$ is ppf, smooth, etale, open embedding, Zariski if f is flat and f_{cl} is so.

We say furthermore f is a flat, smoooth, etale, Zariski covering if f_{cl} is moreover surjective.

Definition 1.1.4.3. We say a morphism of prestacks $f : \mathcal{X} \to \mathcal{Y}$ is affine schematic, if for any affine scheme $S \to \mathcal{Y}$, we have the pullback $S \times_{\mathcal{Y}} \mathcal{X}$ affine.

Definition 1.1.4.4. We say an affine schematic map of prestack $\mathcal{X} \to \mathcal{Y}$ is flat, ppf, smooth, etale, open, Zariski if the pullback map is so for any $S \to \mathcal{Y}$ where S is affine.

Definition 1.1.4.5. We say a prestack $\mathcal{X}$ satisfies flat, ppf, smooth, etale, Zariski descent if the following conditions holds.
(1) $\mathcal{X}(\varnothing) = *$.
(2) $\mathcal{X}(S_1 \sqcup S_2) = \mathcal{X}(S_1) \times \mathcal{X}(S_2)$.
(3) For $S' \to S$ a covering, the map $\mathcal{X}(S) \to \mathrm{Tot}(\mathcal{X}(S'^{\bullet}/S))$ is an isomorphism.

We define the category **Stk** (we omit the topology in the notation however) to be the full subcategory of **pStk** with objects satisfying the above descent conditions.

Definition 1.1.4.6. We fix a topology $T \in \{\text{Zariski, etale, smooth, ppf, flat, ...}\}$. We define a map $\mathcal{X} \to \mathcal{Y}$ to be a T-equivalence if for any $\mathcal{Z} \in \mathbf{Stk}$ with respect to T, we have an isomorphism $\mathrm{Hom}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{Hom}(\mathcal{X}, \mathcal{Z})$.

Construction 1.1.4.7. The inclusion $\mathbf{Stk} \hookrightarrow \mathbf{pStk}$ admit a left adjoint, which is the localization functor at the T-equivalences. We denote this localization by $L : \mathbf{pStk} \to \mathbf{Stk}$, and we sometimes abuse notations and write $L : \mathbf{pStk} \to \mathbf{Stk} \to \mathbf{pStk}$. We call it the sheafification functor. By construction, it commutes with finite limits.

Lemma 1.1.4.8. We fix a topology $T \in \{\text{Zariski, etale, ppf, flat}\}$. Then, let $f : \mathcal{X} \to \mathcal{Y}$ be an T-surjection. Then, we have

$$|\mathrm{Bar}(\mathcal{X}^{\bullet}/\mathcal{Y})| \to \mathcal{Y}$$

is a T-equivalence.

Proof: By [Lur08] Corollary 6.2.3.5, it suffices to prove that $\tau_{\leqslant -1}(\mathcal{X} \to \mathcal{Y})$ is the final object of $\mathbf{pStk}_{/\mathcal{Y}}$. This is direct by surjectivity. $\square$

Corollary 1.1.4.9. Let $\mathcal{X} \to \mathcal{Y}$ be a T-surjection. Then, we have an equivalence of categories

$$\left|\mathrm{Bar}(\mathbf{Stk}^{\bullet}_{\mathcal{X}}/\mathbf{Stk}_{\mathcal{Y}})\right| = \mathbf{Stk}_{\mathcal{Y}}$$

Proof: By Lemma 1.1.4.8, we have $\mathcal{X} \to \mathcal{Y}$ an effective epimorphism in **Stk**. This then follows from the property that **Stk**, as a localization of **pStk** by a topology, is a topos. $\square$

Corollary 1.1.4.10.

Lemma 1.1.4.11. The image of the Yoneda functor $\mathbf{Aff} \hookrightarrow \mathbf{pStk}$ lies in **Stk**.

Proof: Since $T = \mathrm{flat}$ is the finest topology, it suffices to show that the flat topology is subcanonical. This is done by reduction to the ordinary faithfully flat descent as follows:

Take a faithfully flat map $A \to B$ and a ring R. We wish to prove that

$$\mathrm{Hom}_{\mathbf{CAlg}^{\geqslant 0}}(R, A) = \mathrm{Tot}_{n \geqslant 1}\big(\mathrm{Hom}_{\mathbf{CAlg}^{\geqslant 0}}(R, B^{\otimes_A n})\big)$$

Note $\mathrm{Hom}_{\mathbf{CAlg}^{\geqslant 0}}(R, -)$ preserves limits, it suffices to prove

$$A = \lim_{\Delta}\big(B \rightrightarrows B \otimes_A B \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots\big)$$

We prove that this is an isomorphism on each cohomology. By flatness, we have

$$H^n(B^{\otimes_A m}) = H^n(A) \otimes_{H^0(A)} H^0(B)^{\otimes_{H^0(A)} m}$$

Then, since taking limits commutes with taking cohomology, we have the map

$$H^n(A) \to \lim_{\Delta} H^n(A) \otimes_{H^0(A)} H^0(B)^{\otimes_{H^0(A)} m} = H^n(A)$$

by the classical faithfully flat descent. Hence the Yoneda functor has image in **Stk**. $\square$

Lemma 1.1.4.12. Consider the Cartesian square

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{X} \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

in **Stk** such that $S, S', \mathcal{X}' \in \mathbf{Aff}$. Moreover, assume f is a T-covering. Then, $\mathcal{X} \in \mathbf{Aff}$.

Proof: Assume $S' \to S = \operatorname{Spec} B \to \operatorname{Spec} A$ induced by a faithfully flat map $f : A \to B$. By an analogous proof of Lemma 1.1.4.11, we have

$$\mathbf{Mod}_A = \lim\left(\mathbf{Mod}_B \rightrightarrows \mathbf{Mod}_{B\otimes_A B} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\right)$$

Passing to commutative algebras, we have

$$\mathbf{CAlg}_A = \lim\left(\mathbf{CAlg}_B \rightrightarrows \mathbf{CAlg}_{B\otimes_A B} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\right)$$

Then, assume $\mathcal{X}' = \operatorname{Spec} T$ for some $T \in \mathbf{CAlg}_B$. We define $R \in \mathbf{CAlg}_A$ given by the descent

$$R = \lim\left(S \rightrightarrows S \otimes_B B \otimes_A \otimes B \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\right)$$

where the structure morphisms of the cosimplicial object is given by the pullback map above, essentially because $\mathcal{X}'$ is the pullback of something (not yet known to be affine) over S. We now prove $\mathcal{X} = \operatorname{Spec} R$. This is because $\mathcal{X}$ and $\operatorname{Spec} R$ share the same descent datum, and hence they are the same by Corollary 1.1.4.9. $\square$

Corollary 1.1.4.13. Let $f : \mathcal{X} \to \mathcal{Y}$ be an affine schematic morphism in **pStk**. Then, the sheafification $L(f) : L(\mathcal{X}) \to L(\mathcal{Y})$ is an affine schematic morphism.

Proof: Take any $S \in \mathbf{Aff}$ and $S \to L(\mathcal{Y})$, and we wish to show the product $S \times_{L(\mathcal{Y})} L(\mathcal{X})$ is affine. By Lemma 1.1.4.12, it suffices to show that for a T-surjection $S' \to S$ we have the fiber product $S' \times_{L(\mathcal{Y})} L(\mathcal{X})$ is affine. Note the unit $\mathcal{Y} \to L(\mathcal{Y})$ is T-surjective, so we may choose S' such that the composition $S' \to S \to L(\mathcal{Y})$ factors through $\mathcal{Y}$ Since L commutes with finite limits and S' is affine hence a stack, it suffices to prove $S' \times_{L(\mathcal{Y})} L(\mathcal{X}) = L(S' \times_{\mathcal{Y}} \mathcal{X})$ is affine. However, $S' \times_{\mathcal{Y}} \mathcal{X}$ is affine, so applying L is an equivalence. $\square$

The interaction of sheafification with n-coconnectivity is rather curious.

Definition 1.1.4.14. We denote ${}^{\leqslant n}\mathbf{Stk}$ the full subcategory of ${}^{\leqslant n}\mathbf{pStk}$ that satisfy T-descent in ${}^{\leqslant n}\mathbf{Aff}$. We denote ${}^{\leqslant n}L$ by the corresonding sheafification functor. By definition, the functor $\mathbf{pStk} \to {}^{\leqslant n}\mathbf{pStk}$ sends $\mathbf{Stk}$ to ${}^{\leqslant n}\mathbf{Stk}$, and preserves T-equivalences.

Construction 1.1.4.15 ([Lur17], 6.5.3). Recall that ${}^{\leqslant n}\mathbf{Stk}$ is the category of sheaves over an n-category ${}^{\leqslant n}\mathbf{Aff}$. Hence, the sheafification functor ${}^{\leqslant n}L$ is the composition of $(n+2)$ times plus construction, with ${}^{\leqslant n}L : {}^{\leqslant n}\mathbf{pStk} \to {}^{\leqslant n}\mathbf{Stk} \to {}^{\leqslant n}\mathbf{pStk}$ preserving k-truncated objects.

1.1.5. Schemes and Artin stacks

Definition 1.1.5.1.

Definition 1.1.5.2. We say an Artin stack is quasi-compact if it admit a smooth affine atlas.

We say a morphism of prestacks $\mathcal{X} \to \mathcal{Y}$ is quasi-compact if its pullback along an affine scheme is a quasi-compact Artin stack.

Definition 1.1.5.3. We let ${}^{\leqslant n}\mathbf{Sch}_{\mathrm{lft}} := \mathbf{Sch} \cap {}^{\leqslant n}\mathbf{Stk}_{\mathrm{lft}}$ and $\mathbf{Sch}_{\mathrm{laft}} := \mathbf{Sch} \cap \mathbf{Stk}_{\mathrm{laft}}$, and define ${}^{\leqslant n}\mathbf{Sch}_{\mathrm{ft}}$ and $\mathbf{Sch}_{\mathrm{aft}}$ to be the intersection of these categories with the quasi-compact schemes (Definition 1.1.5.2).

1.2. Quasi-coherent sheaves

1.2.1. Basic properties

Definition 1.2.1.1. We define the category of quasi-coherent sheaves over affine schemes to be a functor

$$\mathrm{QCoh}_{\mathbf{Aff}} : \mathbf{Aff} = \mathbf{CAlg}(\mathbf{Vect}^{\leqslant 0})^{\mathrm{op}} \to \mathbf{DGCat}$$

defined to be the composition

$$\mathbf{CAlg}(\mathbf{Vect}^{\leqslant 0})^{\mathrm{op}} \to \mathbf{CAlg}(\mathbf{Vect})^{\mathrm{op}} \xrightarrow{\mathbf{Mod}} \mathbf{DGCat}$$

where the second functor sends a ring A to the category $\mathbf{Mod}_A$ of its $\mathbb{E}_\infty$-modules, and a map of rings $A \to B$ the restriction functor $\mathbf{Mod}_B \to \mathbf{Mod}_A$. The functor is obviously continuous.

Equivalently, we interpret the above functor as $S \mapsto \mathrm{QCoh}(S)$ for $S \in \mathbf{Aff}$, and the transition functors are given by $f : S \to S'$ inducing $f_* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(S')$. For this reason, we sometimes write the above functor as QCoh_* to indicate that we are using $(-)_*$ for the transitions.

remark 1.2.1.1. In [GR], the authors used the category of $\mathbb{E}_1$-left modules instead. We can see that this is equivalent to the category of $\mathbb{E}_\infty$-modules via the theorem of Lurie ([Lur17], Corollary 4.5.1.5). We remark here that although the category of $\mathbb{E}_1$-left modules admit no tensor structure, the viewpoint of $\mathbb{E}_1$-left modules is also useful, for example, when considering Koszul duality of polynomial algebras.

Definition 1.2.1.2. By passing to left adjoints, we obtain a functor $\mathrm{QCoh}^*_{\mathbf{Aff}} : \mathbf{Aff}^{\mathrm{op}} \to \mathbf{DGCat}$, to be thought of as assigning a morphism $f : S \to S'$ to $f^* : \mathrm{QCoh}(S') \to \mathrm{QCoh}(S)$.

Definition 1.2.1.3. We define $\mathrm{QCoh}^*_{\mathbf{pStk}} : \mathbf{pStk}^{\mathrm{op}} \to \mathbf{DGCat}$ to be the right Kan extension of the functor $\mathrm{QCoh}^*_{\mathbf{Aff}} : \mathbf{Aff}^{\mathrm{op}} \to \mathbf{DGCat}$ along the Yoneda embedding $\mathbf{Aff}^{\mathrm{op}} \to \mathbf{pStk}^{\mathrm{op}}$.

Unwinding the definition of the limit, for any prestack $\mathcal{X}$, we can think an object of $\mathrm{QCoh}(\mathcal{X})$ as assinging each $S \to \mathcal{X}$ an element $\mathcal{F}_S \in \mathrm{QCoh}(S)$ for S affine, and such that $\mathcal{F}_S$ are compatible with pulling back along the transition maps.

Since the limit is taken in $\mathbf{DGCat}$, for $f : \mathcal{X} \to \mathcal{Y}$, there is a functor $f_* : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ right adjoint to $f^* : \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$.

Proposition 1.2.1.4. For $\mathcal{X} \in {}^{\leqslant n}\mathbf{pStk}$, we have

$$\mathrm{QCoh}(\mathcal{X}) = \left(\mathrm{RKE}_{{}^{\leqslant n}\mathbf{Aff}^{\mathrm{op}} \hookrightarrow {}^{\leqslant n}\mathbf{pStk}^{\mathrm{op}}} \mathrm{QCoh}\right)(\mathcal{X})$$

Proof: By the cofinality obvious. □

Proposition 1.2.1.5. For $\mathcal{X} \in {}^{\leqslant n}\mathbf{pStk}_{\mathrm{lft}}$, we have

$$\mathrm{QCoh}(\mathcal{X}) = \left(\mathrm{RKE}_{{}^{\leqslant n}\mathbf{Aff}^{\mathrm{op}}_{\mathrm{ft}} \hookrightarrow {}^{\leqslant n}\mathbf{pStk}^{\mathrm{op}}_{\mathrm{ft}}} \mathrm{QCoh}\right)(\mathcal{X})$$

Proof: Again by the cofinality obvious. □

Construction 1.2.1.6. However, the functor $\mathrm{QCoh}^* : \mathbf{pStk}^{\mathrm{op}} \to \mathbf{DGCat}$ is not convergent. Take $S = \operatorname{Spec} k[\eta]$ with η in cohomological degree -2. Then, we have $k[\eta]$ the free $\mathbb{E}_1$-algebra under 1 generator in degree -2. Thus, an $k[\eta]$-module is simply a k-module equipped with a degree -2 endomorphism. However, taking the limit gives a k-module equipped with a *nilpotent* degree -2 endomorphism.

todo 1.2.1.1. After finishing descent, finish the descent for quasi-coherent sheaves

We now construct a t-structure on $\mathrm{QCoh}(\mathcal{X})$.

Construction 1.2.1.7. Recall we have the theorem of Lurie ([Lur17], Proposition 1.2.1.16) that a t-structure on a stable ∞-category $\mathcal{C}$ is equivalent to the data of a localization functor $\tau^{\leqslant 0} : \mathcal{C} \to \mathcal{C}$ subject to certain conditions.

We first recall that for $S \in \mathbf{Aff}$, $\mathrm{QCoh}(S)$ inherits a natural t-structure inherited from the forgetful functor $\mathrm{QCoh}(S) \to \mathbf{Vect}_k$. Notice that the pullback functors over affine schemes are right t-exact, for the rings are all connective. Then, by the description of limit categories by Cartesian sections, we see that colimits in such categories are computed pointwise. In particular, since $\mathrm{QCoh}(S)^{\leqslant 0}$ is closed under colimits, the subcategory

$$\mathrm{QCoh}(\mathcal{X})^{\leqslant 0} \subset \mathrm{QCoh}(\mathcal{X})$$

spanned by object with $(\mathcal{F}_S)$ satisfying $\mathcal{F}_S \in \mathrm{QCoh}(S)^{\leqslant 0}$ for any S is closed under colimits. Thus, we have its right adjoint $\tau^{\leqslant 0} : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{X})^{\leqslant 0}$. Then by verifying the Ext condition of the localization functor $\tau^{\geqslant 1} = \mathrm{cofib}(\tau^{\leqslant 0} \to \mathrm{id})$, we obtain a t-structure on $\mathrm{QCoh}(\mathcal{X})$.

We have constructed the pushforward functor as above. In partiuclar, we notice that it is discontinuous, as it arise from a general right adjoint of a functor in $\mathbf{Pr}^L_{\mathrm{St}}$. However, there are some nice properties which we can deduce from and used to deduce the continuity of the pushforward functor. Namely, we have the following.

Proposition 1.2.1.8. Let $f : \mathcal{X}_1 \to \mathcal{X}_2$ be a morphism of prestacks, and consider the pullback diagram

$$\begin{tikzcd} \mathcal{Y}_1 \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{X}_1 \arrow[d, "f"] \\ \mathcal{Y}_2 \arrow[r, "g"'] & \mathcal{X}_2 \end{tikzcd}$$

Then, if f is schematic and quasi-compact (recall the definitions in Definition 1.1.4.3 and Definition 1.1.5.2), we have the following property.
(1) $f_* : \mathrm{QCoh}(\mathcal{X}_1) \to \mathrm{QCoh}(\mathcal{X}_2)$ is continuous.
(2) The base change morphism $g^* \circ f_* \to f'_* \circ g'^*$ is an isomorphism.

Proof:

□

1.2.2. The symmetric monoidal structure of QCoh

We aim to give a right lax symmetric monoidal structure on QCoh^* in this subsection. Before this, we first prove a baby version of the theorem.

Proposition 1.2.2.1. Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathbf{Pr}^L_{\mathrm{St}}$, and let T_1, T_2 be monads over $\mathcal{C}_1, \mathcal{C}_2$ respectively. Then, we have the functor $\mathbf{Alg}_{T_1}(\mathcal{C}_1) \otimes \mathbf{Alg}_{T_2}(\mathcal{C}_2) \to \mathbf{Alg}_{T_1 \otimes T_2}(\mathcal{C}_1 \otimes \mathcal{C}_2)$ an equivalence.

Proof: Notice that the forgetful functors U_{T_1} and U_{T_2} preserves geometric realizations, thus so does $U_{T_1} \otimes U_{T_2}$. By the Barr-Beck-Lurie theorem, to show

$$F_{T_1} \otimes F_{T_2} : \mathcal{C}_1 \otimes \mathcal{C}_2 \to \mathbf{Alg}_{T_1}(\mathcal{C}_1) \otimes \mathbf{Alg}_{T_2}(\mathcal{C}_2)$$

is a monadic left adjoint, it suffices to show $U_{T_1} \otimes U_{T_2}$ is conservative. This is because $\mathcal{C}_1 \otimes \mathcal{C}_2$ is generated by the image of $\mathcal{C}_1 \times \mathcal{C}_2$, by taking a presentation of $\mathcal{C}_i$ by sites.

Then, the monads induced by $F_{T_1} \otimes F_{T_2}$ and $F_{T_1 \otimes T_2}$ clearly coincide, thus identifying the corresponding algebra categories. □

Corollary 1.2.2.2. The functor $\mathrm{QCoh}^*_{\mathbf{Aff}} : \mathbf{Aff}^{\mathrm{op}} \to \mathbf{DGCat}$ admit a canonical symmetric monoidal structure, where $\mathbf{Aff}^{\mathrm{op}} \simeq \mathbf{CAlg}(\mathbf{Vect}^{\leqslant 0})$ is equipped with the coCartesian monoidal structure.

Construction 1.2.2.3. We now construct a right lax symmetric monoidal structure on the functor $\mathrm{QCoh}^* : \mathbf{pStk}^{\mathrm{op}} \to \mathbf{DGCat}$ using Corollary 1.2.2.2. Consider the presentation

$$\mathrm{QCoh}(\mathcal{X}) \otimes \mathrm{QCoh}(\mathcal{Y}) = \left(\lim_{S \to \mathcal{X}} \mathrm{QCoh}(S)\right) \otimes \left(\lim_{T \to \mathcal{Y}} \mathrm{QCoh}(T)\right)$$

and

$$\mathrm{QCoh}(\mathcal{X} \times \mathcal{Y}) = \lim_{S \to \mathcal{X} \times \mathcal{Y}} \mathrm{QCoh}(S)$$

However, the functor $\mathbf{Aff}_{/\mathcal{X}}^{\mathrm{op}} \times \mathbf{Aff}_{/\mathcal{Y}}^{\mathrm{op}} \to \mathbf{Aff}_{/\mathcal{X} \times \mathcal{Y}}^{\mathrm{op}}$ is cofinal, so

$$\mathrm{QCoh}(\mathcal{X} \times \mathcal{Y}) = \lim_{S \times T \to \mathcal{X} \times \mathcal{Y}} \mathrm{QCoh}(S) \otimes \mathrm{QCoh}(T)$$

and the desired $\mathrm{QCoh}(\mathcal{X}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X} \times \mathcal{Y})$ is then given by exchanging limits with the tensor product.

We denote the functor $\mathrm{QCoh}(\mathcal{X}) \otimes \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X} \times \mathcal{Y})$ by the exterior tensor, whose precomposition with $\mathrm{QCoh}(\mathcal{X}) \times \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X}) \otimes \mathrm{QCoh}(\mathcal{Y})$ denoted by $\mathcal{F}_1 \otimes \mathcal{F}_2 \mapsto \mathcal{F}_1 \boxtimes \mathcal{F}_2$.

We also call the exterior tensor map the categorical Künneth formula map, and say the categorical Künneth formula holds if the above map is an equivalence of categories.

Corollary 1.2.2.4. Since taking tensor products with dualizable categories commutes with taking limits, the categorical Künneth formula holds for the stacks where one has QCoh dualizable.

Construction 1.2.2.5. We now apply Construction 1.2.2.3 to give a symmetric monoidal structure on the category $\mathrm{QCoh}(\mathcal{X})$ for any prestack $\mathcal{X}$. We define

$$\mathcal{F}_1 \otimes \mathcal{F}_2 = \Delta_{\mathcal{X}}^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

Thus, the pullback functor is naturally symmetric monoidal. The pushforward functor, as its right adjoint, is naturally right lax symmetric monoidal. This in particular yields a projection formula map

$$\mathcal{F} \otimes f_*(\mathcal{G}) \to f_*(f^*(\mathcal{F}) \otimes \mathcal{G})$$

for any $f : \mathcal{X} \to \mathcal{Y}$ and any $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ and $\mathcal{G} \in \mathrm{QCoh}(\mathcal{X})$. We say the projection formula holds if the above map is an isomorphism.

Proposition 1.2.2.6. The projection formula for QCoh holds for affine schematic morphisms $f : \mathcal{X} \to \mathcal{Y}$.

Proof:

□

todo 1.2.2.1. Finish the proofs of the properties of affine schematic morphisms.

We then consider a stronger version of the categorical Künneth formula for QCoh. Namely, we consider when the following map is an equivalence.

$$\mathrm{QCoh}(\mathcal{X}) \otimes_{\mathrm{QCoh}(\mathcal{Z})} \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$$

It turns out that this is related to the rigidity of QCoh.

todo 1.2.2.2. Finish this part.

1.3. Ind-coherent sheaves

We then move on to the topic of ind-coherent sheaves. For a brief summary, recall that we have that for smooth classical schemes, the coherent sheaves can be identified with the perfect sheaves (Lemma 1.3.1.4). However, when S is singular, the two categories differ, however slightly. We have that they always agree on the eventually coconnective parts (todo 2.1). This suggests that coherent sheaves can detect the singularities of the scheme. Then, the category of coherent sheaves is not presentable, since it does not contain infinite direct sums. The way to resolve this is to consider the ind-coherent sheaves, which is a certain ind-completion of coherent sheaves. Going back to the topic of singularities, we can define the singular support of ind-coherent sheaves, which contains the data of where this ind-coherent sheaf is non-perfect. In particular, the quasi-coherent sheaves can be identified with the ind-coherent sheaves with zero singular support.

A second reason for considering ind-coherent sheaves is that the functoriality for ind-coherent sheaves are better than the ones for quasi-coherent sheaves. We will see this in todo 2.1.

1.3.1. Definitions

Definition 1.3.1.1. For $X \in \mathbf{Sch}_{\mathrm{aft}}$, we define $\mathrm{Coh}(X)$ to be the category of bounded complexes with cohomologies lying in $\mathrm{Coh}(X_{\mathrm{cl}})$. We define $\mathrm{IndCoh}(X)$ to be the ind-completion $\mathrm{Ind}(\mathrm{Coh}(X))$.

On the level of affine schemes, for $A \in \mathbf{CAlg}^{\leqslant 0}_{\mathrm{aft}}$ we have $\mathrm{Coh}(\operatorname{Spec} A)$ the subcategory of $\mathbf{Mod}_A$ given by objects M with bounded cohomologies and such that $H^i(M)$ is finitely generated (equivalent to finitely presented) as an $H^0(A)$-module.

By definition, there is a comparison map $\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ given by ind-extending the inclusion $\mathrm{Coh}(X) \hookrightarrow \mathrm{QCoh}(X)$.

Definition 1.3.1.2. Let $A \in \mathbf{CAlg}^{\leqslant 0}_{\mathrm{aft}}$. We say $M \in \mathbf{Mod}_A$ is almost perfect if $M \in \mathbf{Mod}_A^{\leqslant k}$ for some k, and $\tau^{\geqslant n}(M)$ is compact in $\mathbf{Mod}_A^{[n,k]}$ for all n.

Theorem 1.3.1.3 ([Lur17], Proposition 7.2.4.17). Let $A \in \mathbf{CAlg}^{\leqslant 0}_{\mathrm{aft}}$ and $M \in \mathbf{Mod}_A$. Then, M is almost perfect iff the following conditions holds:
(1) $M \in \mathbf{Mod}_A^{-}$. That is, $M \in \mathbf{Mod}_A^{\leqslant k}$ for some k.
(2) For every n we have $H^n(M)$ finitely generated as an $H^0(A)$-module.

Thus, a coherent module is by definition an almost perfect module which is also eventually coconnective. We will use this later on.

Lemma 1.3.1.4. When X is a smooth classical scheme finite type over a field, Ψ_X is an equivalence of categories.

Proof: It suffices to prove that $\mathrm{Coh}(X) = \mathrm{Perf}(X)$. Notice that since X is Noetherian, the property of coherence and perfectness is local. Hence, we may assume that $X = \operatorname{Spec} A$ where A is a Noetherian regular ring of finite dimension. Hence, A is of finite global dimension. Thus every finitely generated (i.e. coherent) A-module admit a finite length resolution. Thus we have obtained the result. $\square$

Definition 1.3.1.5. We say X is eventually coconnective if $\mathcal{O}_X \in \mathrm{Coh}(X)$.

Lemma 1.3.1.6. For eventually coconnective X, the functor Ψ_X admits a fully faithful left adjoint, which we will denote by Ξ_X.

Proof: If $\mathcal{O}_X \in \mathrm{Coh}(X)$, then we have a fully faithful functor $\mathrm{Perf}(X) \hookrightarrow \mathrm{Coh}(X)$. Then, for such schemes, we have $\mathrm{QCoh}(X) = \mathrm{Ind}(\mathrm{Perf}(X))$, so Ξ_X is obtained by ind-extending the inclusion. Hence, $\Psi_X \circ \Xi_X$ is the ind-extension of the inclusion $\mathrm{Perf}(X) \hookrightarrow \mathrm{QCoh}(X)$, which is clearly the identity. The left adjoint property is obvious. $\square$

We now construct a t-structure on the category $\mathrm{IndCoh}(X)$.

Lemma 1.3.1.7. Let $\mathcal{C}_0 \in \mathbf{DGCat}^{\text{non-cocmpl}}$ be a non-cocomplete DG category, endowed with a t-structure. Then, we have $\mathcal{C} = \mathrm{Ind}(\mathcal{C}_0) \in \mathbf{DGCat}$ admit a natural t-structure, with the following properties:

(1) The inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is t-exact.
(2) The subcategories $\mathcal{C}^{\leqslant 0}$ and $\mathcal{C}^{\geqslant 0}$ are compactly generated under filtered colimits by $\mathcal{C}_0^{\leqslant 0}$ and $\mathcal{C}_0^{\geqslant 0}$.
(3) Let $\mathcal{D}$ be another DG category with a t-structure, such that $\mathcal{D}^{\geqslant 0}$ is closed under filtered colimits. Then, a continuous functor $F : \mathcal{C} \to \mathcal{D}$ is t-exact iff $F|_{\mathcal{C}_0}$ is.

Proof: The existence of the t-structure is an application of [Lur17], Proposition 1.2.1.16. For (3), if $F|_{\mathcal{C}_0}$ is t-exact, we clearly have F preserves both connective and coconnective objects, since both are closed under taking filtered colimits. $\square$

Via Lemma 1.3.1.7, we may define a t-structure on $\mathrm{IndCoh}(X)$.

Corollary 1.3.1.8. The t-structure on $\mathrm{IndCoh}(X)$ makes Ψ_X t-exact, and is compatible with filtered colimits.

Proposition 1.3.1.9. The functor $\Psi_X : \mathrm{IndCoh}\,(X)^{\geqslant 0} \to \mathrm{QCoh}(X)^{\geqslant 0}$ is an equivalence of categories. Hence, we have $\mathrm{IndCoh}(X)^{+} = \mathrm{QCoh}(X)^{+}$.

Proof: It suffices to show that $\mathrm{QCoh}(X)^{\geqslant 0}$ is compactly generated by $\mathrm{Coh}(X)^{\geqslant 0}$ under filtered colimits. This is done by first passing to modules, and use that coherent sheaves are almost perfect. $\square$

Lemma 1.3.1.10. We have $\mathrm{Coh}(X) = \mathrm{IndCoh}(X)^{\omega}$.

Proof: Firstly note that $\mathrm{IndCoh}(X)^\omega$ is the Karoubi completion of $\mathrm{Coh}(X)$. Then, for $\mathcal{F} \in \mathrm{IndCoh}(X)^\omega$, assume it is a direct summand (retract) of $\mathcal{F}' \in \mathrm{Coh}(X)$. Then we have $\Psi_X(\mathcal{F})$ is a direct summand of $\Psi_X(\mathcal{F}')$. But since $\mathrm{Coh}(X)$ is Karoubi complete in $\mathrm{QCoh}(X)$, we may assume $\Psi_X(\mathcal{F}) = \Psi_X(\mathcal{F}')$. Then by definition, $\mathcal{F}, \mathcal{F}' \in \mathrm{IndCoh}(X)^\omega \subset \mathrm{IndCoh}(X)^+ \simeq \mathrm{QCoh}(X)^+$, we have $\mathcal{F} = \mathcal{F}'$. $\square$

Construction 1.3.1.11. We now construct an action of $\mathrm{QCoh}(X)$ on $\mathrm{IndCoh}(X)$, such that $\Psi_X : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ is a map of $\mathrm{QCoh}(X)$-modules. For the action, notice that $\mathrm{Perf}(X) \subset \mathrm{Coh}(X)$ gives an action, whose ind-extension is the desired action. Moreover, we claim that this action is unique because the action of $\mathrm{Perf}(X)$ on $\mathrm{IndCoh}(X)$ must preserves compact objects, which is by dualizability of $\mathrm{Perf}(X)$.

Lemma 1.3.1.12. Suppose $X \in \mathbf{Sch}_{\mathrm{aft}}$ and $\mathcal{E} \in \mathrm{QCoh}(X)^b$, with $\mathcal{E} \otimes - : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ left t-exact up to finite shift. Then, the functor

$$\mathcal{E} \otimes - : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X)$$

is also left t-exact up to finite shift.

We temporarily postpone the proof (here) of this lemma to the following section, after we have developed certain functorialities of IndCoh.

1.3.2. The usual pushforward and pullback functors

Unlike the QCoh, we need to construct the pullback and pushforward functors by ourselves, and prove that they satisfy a good functoriality.

Construction 1.3.2.1. Let $f : X \to Y$ be a morphism in $\mathbf{Sch}_{\mathrm{aft}}$. We construct a functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y) \in \mathbf{DGCat}$$

making the following diagram commute:

$$\begin{tikzcd} \mathrm{IndCoh}(X) \arrow[r, "\Psi_X"] \arrow[d, "f_*^{\mathrm{IndCoh}}"'] & \mathrm{QCoh}(X) \arrow[d, "f_*"] \\ \mathrm{IndCoh}(Y) \arrow[r, "\Psi_Y"'] & \mathrm{QCoh}(Y) \end{tikzcd}$$

Moreover, we can show that the constructed functor f_*^{IndCoh} is left t-exact.

By definition, functors in $\mathbf{DGCat}$ must be continuous, so f_*^{IndCoh} is the ind-extension of the restriction to $\mathrm{Coh}(X)$. Then, the above commutative diagram is nothing but

$$\Psi_Y \circ f_*^{\mathrm{IndCoh}}|_{\mathrm{Coh}(X)} = f_*|_{\mathrm{Coh}(X)}$$

Since $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is left t-exact and $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)^+$, we have the image of the functor $\Psi_Y \circ f_*^{\mathrm{IndCoh}}$ contained in $\mathrm{QCoh}(Y)^+$. Hence, we may define $f_*^{\mathrm{IndCoh}}|_{\mathrm{Coh}(X)}$ by

$$f_*^{\mathrm{IndCoh}}|_{\mathrm{Coh}(X)} = \Psi_Y|^{-1}_{\mathrm{IndCoh}(Y)^+} \circ f_*|_{\mathrm{Coh}(X)}$$

and finally define f_*^{IndCoh} by ind-extending the above functor.

Moreover, we have $f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ is a functor of $\mathrm{QCoh}(Y)$-modules.

Proposition 1.3.2.2. The above construction upgrades to a functor

$$\mathrm{IndCoh}_{\mathbf{Sch}_{\mathrm{aft}}} : \mathbf{Sch}_{\mathrm{aft}} \to \mathbf{DGCat}$$

equipped with a natural transformation

$$\Psi : \mathrm{IndCoh}_{\mathbf{Sch}_{\mathrm{aft}}} \to \mathrm{QCoh}_{\mathbf{Sch}_{\mathrm{aft}}}$$

such that at the level of objects and 1-morphisms we obtain the above construction.

The construction of the full functorial structure is not direct because we need to specify the higher structures. However, we do not need to determine them by force, but via a clever construction of an auxillary ∞-category. We will use similar techniques to obtain further functorialities of IndCoh.

Definition 1.3.2.3. We define a 1-fully faithful functor of ∞-categories to be a functor $F : \mathcal{C} \to \mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{C}}(c_1, c_2) \to \mathrm{Hom}_{\mathcal{D}}(F(c_1), F(c_2))$ is an inclusion of connected components of spaces. In this case, we say $\mathcal{C}$ is a 1-full subcategory of $\mathcal{D}$.

We first give a precise formulation of our current status of the functoriality to be constructed.

Definition 1.3.2.4. Let $\mathcal{I}$ and $\mathcal{C}$ be ∞-categories. We define an assignment F from $\mathcal{I}$ to $\mathcal{C}$ to be a map of sets $\pi_0(\mathcal{I}^{\simeq}) \to \pi_0(\mathcal{C}^{\simeq})$, denoted by $i \rightsquigarrow F(i)$. We say it can be extended to a functor if there is a functor $F' : \mathcal{I} \to \mathcal{C}$ such that $F'(i) = F(i)$ for all $i \in \pi_0(\mathcal{I}^{\simeq})$. We abuse notations and denote this functor F' also by F.

Lemma 1.3.2.5. Let $T : \mathcal{C} \to \mathcal{D}$ be a 1-fully faithful functor of ∞-categories. Let $\mathcal{I}$ be another ∞-category, and let $i \rightsquigarrow F'(i)$ be an assignment from $\mathcal{I}$ to $\mathcal{C}$ such that $i \rightsquigarrow T \circ F'(i)$ can be extended to a functor $F : \mathcal{I} \to \mathcal{D}$. Then, suppose that for every $\alpha \in \mathrm{Hom}_{\mathcal{I}}(i_1, i_2)$, $F(\alpha) \in \mathrm{Hom}_{\mathcal{D}}(F(i_1), F(i_2))$ lies in the image of T. Then, there exists a unique extension of F' to a functor.

Proof: This is obvious. □

Lemma 1.3.2.6. We keep the settings as in the previous lemma. Now, let F_1' and F_2' be two functors from $\mathcal{I}$ to $\mathcal{C}$. Let $i \rightsquigarrow \psi_i' \in \mathrm{Hom}_{\mathcal{C}}(F_1'(i), F_2'(i))$ be an assignment from $\mathcal{I}$ to $\mathrm{Fun}(\Delta^1, \mathcal{C})$. Assume $i \rightsquigarrow T(\psi_i') \in \mathrm{Hom}_{\mathcal{D}}(F_1(i), F_2(i))$ can be extended to a functor $\psi : \mathcal{I} \to \mathrm{Fun}(\Delta^1, \mathcal{D})$, then there exists a unique extension of ψ' to a functor $\psi' : \mathcal{I} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$.

Proof: Also obvious. □

Construction 1.3.2.7. We define the category $\mathbf{DGCat}^+$ to be the 1-full subcategory of $\mathbf{DGCat}^{\text{non-cocmpl}}$, spanned by objects that are non-cocomplete DG categories $\mathcal{C}$ with the following properties:
(1) It is equipped with a t-structure, and $\mathcal{C} = \mathcal{C}^+$.
(2) $\mathcal{C}^{\geqslant 0}$ has all filtered colimits and the inclusion $\mathcal{C}^{\geqslant 0} \hookrightarrow \mathcal{C}$ commutes with filtered colimits.

We take the 1-morphisms to be spanned by the functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ left t-exact up to a finite shift, and $F|_{\mathcal{C}_1^{\geqslant 0}}$ preserves filtered colimits.

We define the category $\mathbf{DGCat}^t$ to be the 1-full subcategory of $\mathbf{DGCat}$, spanned by the objects that are DG categories $\mathcal{C}$ with the following properties:
(1) $\mathcal{C}$ is equipped with a t-structure, and is compactly generated by $\mathcal{C}^+$.
(2) $\mathcal{C}^{\geqslant 0}$ admits all filtered colimits.

We take the 1-morphisms to be spanned by the continuous functors left t-exact up to a finite shift.

Moreover, we can identify $\mathbf{DGCat}^t$ as a 1-full subcategory of $\mathbf{DGCat}^+$ by $\mathcal{C} \mapsto \mathcal{C}^+$.

Proof: We now prove Proposition 1.3.2.2. We set $\mathcal{I} = \mathbf{Sch}_{\text{aft}}$, $\mathcal{C} = \mathbf{DGCat}^t$, $\mathcal{D} = \mathbf{DGCat}$, $F = \mathrm{QCoh}_{\mathbf{Sch}_{\text{aft}}} : \mathbf{Sch}_{\text{aft}} \to \mathbf{DGCat}$, and the assignment F' to be $X \rightsquigarrow \mathrm{QCoh}(X) \in \mathbf{DGCat}^t$. Now, apply Lemma 1.3.2.5, we obtain a genuine functor $\mathrm{QCoh}^t_{\mathbf{Sch}_{\text{aft}}} : \mathbf{Sch}_{\text{aft}} \to \mathbf{DGCat}^t$. Then, we may compose into $\mathrm{QCoh}^+_{\mathbf{Sch}_{\text{aft}}} : \mathbf{Sch}_{\text{aft}} \to \mathbf{DGCat}^+$. We now set $\mathcal{C} = \mathbf{DGCat}^t$, $\mathcal{D} = \mathbf{DGCat}^+$, $F = \mathrm{QCoh}^+_{\mathbf{Sch}_{\text{aft}}}$, and the assignment $F' : X \rightsquigarrow \mathrm{IndCoh}(X) \in \mathbf{DGCat}^t$. Thus, again by Lemma 1.3.2.5, we obtain an extension $\mathrm{IndCoh}^t_{\mathbf{Sch}_{\text{aft}}} : \mathbf{Sch}_{\text{aft}} \to \mathbf{DGCat}^t$. Post-composing with the 1-fully faithful inclusion $\mathbf{DGCat}^t \hookrightarrow \mathbf{DGCat}$, we obtain the desired lift as a functor $\mathrm{IndCoh}_{\mathbf{Sch}_{\text{aft}}} : \mathbf{Sch}_{\text{aft}} \to \mathbf{DGCat}$. The natural transformation Ψ then is obtained by applying Lemma 1.3.2.6 to the obvious assignment, and the naturality condition used is ensured by Construction 1.3.2.1. $\square$

We are not yet ready to define the category of ind-coherent sheaves for general prestacks. We will first construct a functor of $*$-pullbacks of ind-coherent sheaves for certain nice morphisms.

Definition 1.3.2.8. Let $f : X \to Y \in \mathbf{Sch}_{\text{aft}}$. We say f is eventually coconnective if $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ sends $\mathrm{Coh}(Y)$ to $\mathrm{Coh}(X)$. The definition will remain unchanged if we replace $\mathrm{Coh}(X)$ above by $\mathrm{QCoh}(Y)^+$.

Lemma 1.3.2.9. f is eventually coconnective iff f^* is of finite tor-amplitude (as the definition Definition 1.1.1.2, and note f^* is naturally right t-exact, so here we use only left t-exactness up to a shift).

Proof: Obviously we have finite tor-amplitude implies eventual coconnectivity. For the converse, since the problem is Zariski local, we may factor f as $X \hookrightarrow Y \times \mathbb{A}^n \to Y$ by almost finite type condition. Then f is of finite tor-amplitude iff $X \hookrightarrow Y \times \mathbb{A}^n$ is, so we may assume f is a closed embedding.

Then, to test that f^* is of finite tor-amplitude, we compute $f^*\big(\mathrm{Coh}(Y)^\heartsuit\big)$. But then, since $\mathrm{Coh}(Y)^\heartsuit = \mathrm{Coh}(Y_{\text{cl}})^\heartsuit$, we may assume Y is itself classical. Then since f is eventually coconnective, so is X. Now, since $f : X \to Y$ is affine schematic, so the projection formula holds, for $\mathcal{F} \in \mathrm{QCoh}(Y)$, we have

$$f_*f^*(\mathcal{F}) = f_*(\mathcal{O}_X) \otimes \mathcal{F}$$

But then f is a closed immersion so affine, and hence f_* is t-exact and conservative. So to prove f^* is of finite tor-amplitude, it suffices to prove $f_*(\mathcal{O}_X)$ is of finite tor-amplitude.

Now, notice that f also satisfy base change, so for any geometric point $s \hookrightarrow Y$, we have $s^*f_*(\mathcal{O}_X) = R\Gamma\left(\mathcal{O}_{X_s}\right)$ is eventually coconnective. Then, we prove that this implies $f_*(\mathcal{O}_X)$ is perfect. Let $E = f_*(\mathcal{O}_X)$. Suppose $s^*(E) \in \mathbf{Mod}_K^{[-n,m]}$, where $K = \mathcal{O}_s(s)$. We prove that on a neighborhood of s, $E \otimes -$ has amplitude $[-n, m]$. Firstly by shifting, we may assume $m = 0$. Then, we prove by induction.

The base case is $n = -1$, which says $s^*E = 0$. By derived Nakayama's lemma, we have the restriction of E to $\operatorname{Spec} \mathcal{O}_{Y,s}$ is 0. Hence, there is a neighborhood of s such that $E = 0$.

Then, we assume the induction hypothesis. We choose a finite rank free sheaf $P \to E$ inducing an isomorphism $H^0(s^*P) \overset{\simeq}{\to} H^0(s^*E)$. We let $F = \operatorname{fib}(P \to E)$. Hence, by the long exact sequence, we have $s^*F \in \mathbf{Mod}_K^{[-n+1,0]}$. Hence, F is of tor-amplitude $[-n+1, 0]$ on some neighborhood of s. Then on the same neighborhood, E is of tor-amplitude $[-n, 0]$. Finally, since Y is quasi-compact, we may find a uniform bound on the tor-amplitude of E. □

remark 1.3.2.1. Recall by [Sta26] 0658, being perfect is equivalent to being finite tor-amplitude and pseudo-coherent. So in fact, the sheaves appearing in the above proof are all perfect.

Definition 1.3.2.10. For $f : X \to Y$ eventually coconnective, we may ind-extend the functor $f^* : \operatorname{Coh}(Y) \to \operatorname{Coh}(X)$ and obtain $f^{\operatorname{IndCoh},*}$. In particular, it commutes with Ψ_X and f^* on coherent sheaves.

Lemma 1.3.2.11. $f^{\operatorname{IndCoh},*}$ is left adjoint to $f_*^{\operatorname{IndCoh}}$.

Proof: Obviously we have for $\mathcal{F}_X \in \operatorname{Coh}(X)$ and $\mathcal{F}_Y \in \operatorname{Coh}(Y)$ an equivalence

$$\begin{aligned}
&\operatorname{Hom}_{\operatorname{IndCoh}(X)}\left(f^{\operatorname{IndCoh},*}(\mathcal{F}_Y), \mathcal{F}_X\right) = \operatorname{Hom}_{\operatorname{QCoh}(X)}(f^*(\mathcal{F}_Y), \mathcal{F}_X) \\
&= \operatorname{Hom}_{\operatorname{QCoh}(Y)}(\mathcal{F}_Y, f_*(\mathcal{F}_{\mathcal{X}})) = \operatorname{Hom}_{\operatorname{IndCoh}(Y)}\left(\mathcal{F}_Y, f_*^{\operatorname{IndCoh}}(\mathcal{F}_X)\right)
\end{aligned}$$

Then writing every object of IndCoh into filtered colimits of these compact generators, we are done. □

Corollary 1.3.2.12. $f^{\operatorname{IndCoh},*} : \operatorname{IndCoh}(Y) \to \operatorname{IndCoh}(X)$ is a map of $\operatorname{QCoh}(Y)$-modules.

Proof: Ind-extend the action of $\operatorname{QCoh}^\omega$ on Coh. □

Proposition 1.3.2.13. $f_*^{\operatorname{IndCoh}} : \operatorname{IndCoh}(X) \to \operatorname{IndCoh}(Y)$ admit a left adjoint iff f is eventually coconnective.

Proof: Let $f^{\mathrm{IndCoh},L}$ denote the left adjoint. Since it preserves all colimits, it preserves compact objects. We let f^L denote the restriction $f^L : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$, and $f^{\mathrm{IndCoh},L}$ is its ind-extension. Hence, it suffices to prove that ??????? □

Corollary 1.3.2.14. By the same process as the proof of Proposition 1.3.2.2, we have the assignment $X \rightsquigarrow \mathrm{IndCoh}(X)$, $f \rightsquigarrow f^{\mathrm{IndCoh},*}$ extends to a functor

$$\mathrm{IndCoh}^*_{\mathbf{Sch}_{\mathrm{aft,ev\text{-}coconn}}} : \mathbf{Sch}^{\mathrm{op}}_{\mathrm{aft,ev\text{-}coconn}} \to \mathbf{DGCat}$$

1.3.3. Basic properties of the functors

We now establish some basic properties of the above constructed functors. We begin with the base change theorem.

Proposition 1.3.3.1. Consider the following Cartesian diagram in $\mathbf{Sch}_{\mathrm{aft}}$:

$$\begin{tikzcd} X_1 \arrow[r, "g_X"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\ulcorner", very near start] & X_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "g_Y"'] & Y_2 \end{tikzcd}$$

Assume f_2 is eventually coconnective. Then, the base change morphism

$$f_2^{\mathrm{IndCoh},*} \circ g_{Y,*}^{\mathrm{IndCoh}} \to g_{X,*}^{\mathrm{IndCoh}} \circ f_1^{\mathrm{IndCoh},*}$$

is an equivalence

Proof: It suffices to show that this is an equivalence for $\mathcal{F} \in \mathrm{Coh}(Y_1)$. In this case, both sides belong to $\mathrm{IndCoh}(X_2)^+$. Thus, it suffices to show that the above map is an equivalence after applying Ψ_{X_2}. Using compatibility of Ψ's with f_*^{IndCoh} and $f^{\mathrm{IndCoh},*}$, this now follows from Proposition 1.2.1.8. □

Corollary 1.3.3.2. The following version of projection formula holds. In this case, we do not need f even to be eventually coconnective.

$$\mathcal{E}_Y \otimes f_*^{\mathrm{IndCoh}} \mathcal{F}_X \xrightarrow{\simeq} f_*^{\mathrm{IndCoh}}(f^*(\mathcal{E}_Y) \otimes \mathcal{F}_X)$$

Proof: Follows from $f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ is a morphism of $\mathrm{QCoh}(Y)$-modules, as in Construction 1.3.2.1. □

The projection formula for f_*^{IndCoh} and $f^{\mathrm{IndCoh},*}$ is much more difficult. For these, we need the Zariski descent for IndCoh.

Lemma 1.3.3.3. Suppose $j : X \hookrightarrow Y$ is an open embedding, then j_*^{IndCoh} is fully faithful.

Proof: We show the counit of the adjunction $j^{\mathrm{IndCoh},*} \circ j_*^{\mathrm{IndCoh}} \to \mathrm{id}$ is an equivalence. Note on both sides the functors are continuous, so it suffices to check that for $\mathcal{F} \in \mathrm{Coh}(X)$ we have $j^{*,\mathrm{IndCoh}} \circ j_*^{\mathrm{IndCoh}}(\mathcal{F}) \to \mathcal{F}$ is an equivalence. Since both sides are in IndCoh^+, applying Ψ_X to both sides it suffices to show $j^* \circ j_* \to \mathrm{id}$ is an equivalence, which is obvious. □

Corollary 1.3.3.4. By Lemma 1.3.1.7 and Lemma 1.3.3.3, we have $j^{\mathrm{IndCoh},*}$ t-exact.

Proposition 1.3.3.5. Let $f : U \to X$ be a Zariski cover. Then, the functor

$$\mathrm{IndCoh}(X) \to \mathrm{Tot}(\mathrm{IndCoh}(U^\bullet))$$

is an equivalence.

Proof: By compactness of X, we immediately reduces to the following case:

$$X = U_1 \cup U_2, U_{12} = U_1 \cap U_2$$

we must show

$$\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(U_1) \times_{\mathrm{IndCoh}(U_{12})} \mathrm{IndCoh}(U_2)$$

is an equivalence.

We first construct the right adjoint of this functor. We send the data

$$\left(\mathcal{F}_i \in \mathrm{IndCoh}(U_i), \mathcal{F}_{12} \in \mathrm{IndCoh}(U_{12}), \mathcal{F}_1, j_{12,1}^{\mathrm{IndCoh},*}(\mathcal{F}_1) = \mathcal{F}_{12} = j_{12,2}^{\mathrm{IndCoh},*}(\mathcal{F}_2)\right)$$

to the obvious ind-coherent sheaf

$$\ker\left(j_{1,*}^{\mathrm{IndCoh}}(\mathcal{F}_1) \oplus j_{2,*}^{\mathrm{IndCoh}}(\mathcal{F}_2) \to j_{12,*}^{\mathrm{IndCoh}}(\mathcal{F}_{12})\right)$$

By the base change theorem Proposition 1.3.3.1 and Lemma 1.3.3.3, the composition

$$\mathrm{IndCoh}(U_1) \times_{\mathrm{IndCoh}(U_{12})} \mathrm{IndCoh}(U_2) \to \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(U_1) \times_{\mathrm{IndCoh}(U_{12})} \mathrm{IndCoh}(U_2)$$

is an equivalence. Then, to show that

$$\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(U_1) \times_{\mathrm{IndCoh}(U_{12})} \mathrm{IndCoh}(U_2) \to \mathrm{IndCoh}(X)$$

is an equivalence, it suffices to show that for $\mathcal{F} \in \mathrm{IndCoh}(X)$, we have

$$\ker\left(j_{1,*}^{\mathrm{IndCoh}} \circ j_1^{\mathrm{IndCoh},*}(\mathcal{F}) \oplus j_{2,*}^{\mathrm{IndCoh}} \circ j_2^{\mathrm{IndCoh},*}(\mathcal{F}) \to j_{12,*}^{\mathrm{IndCoh}} \circ j_{12}^{\mathrm{IndCoh},*}(\mathcal{F})\right)$$

is an equivalence. Again, since all functors are continuous, we may assume $\mathcal{F} \in \mathrm{Coh}(X)$, and again both sides are in $\mathrm{IndCoh}\,(X)^+$, so we may apply Ψ_X to both sides. In this case, it reduces to the Zariski descent of quasi-coherent sheaves, which is by todo 2.1. □

Corollary 1.3.3.6. Let $f : U \to X$ be a Zariski cover. Then, $\mathcal{F} \in \mathrm{IndCoh}(X)$ is in $\mathrm{IndCoh}(X)^{\geqslant 0}$ or $\mathrm{IndCoh}(X)^{\leqslant 0}$ iff $f^{\mathrm{IndCoh},*}(\mathcal{F})$ does.

Proof: For only if, we have $j^{\mathrm{IndCoh},*}$ t-exact by Corollary 1.3.3.4. For the other direction, assume $f^{\mathrm{IndCoh},*}(\mathcal{F}) \in \mathrm{IndCoh}(U)^{\leqslant 0}$, apply Ψ_U, and we obtain the result from todo 2.1. Assume

$f^{\mathrm{IndCoh},*}(\mathcal{F}) \in \mathrm{IndCoh}(U)^{\geqslant 0}$, the assertion follows from the construction of the inverse functor as in Proposition 1.3.3.5. $\square$

Proof: We may now prove Lemma 1.3.1.12. By Lemma 1.3.1.7, it suffices to prove that $\mathcal{E} \otimes -$ sends $\mathrm{Coh}(X)^{\geqslant 0}$ to $\mathrm{IndCoh}(X)^{\geqslant -n}$. To show this, it suffices to prove $\mathrm{Coh}(X)^{\heartsuit}$ is mapped to $\mathrm{IndCoh}(X)^{\geqslant -n}$. We consider the closed embedding $i : X_{\mathrm{cl}} \hookrightarrow X$. Note $i_*^{\mathrm{IndCoh}} = i_* : \mathrm{Coh}(X_{\mathrm{cl}})^{\heartsuit} \to \mathrm{Coh}(X)^{\heartsuit}$ induces an equivalence of categories. By the projection formula above, we have

$$\mathcal{E} \otimes i_*^{\mathrm{IndCoh}}(\mathcal{F}') \simeq i_*^{\mathrm{IndCoh}}(i^*(\mathcal{E}) \otimes \mathcal{F}')$$

and notice that $i^*(\mathcal{E})$ is bounded of the same tor amplitude as $\mathcal{E}$, and i_*^{IndCoh} is t-exact, we may assume that X is classical. Moreover, by Corollary 1.3.3.6, we may assume that X is affine. In this case, the assumption that $\mathcal{E}$ is bounded of tor amplitude $[-n, 0]$ is equivalent that $\mathcal{E}$ has a flat resolution of length at most n. Again by cohomological devissage, it suffices to prove for $\mathcal{E}$ a flat module. In this case, $\mathcal{E} \otimes - : \mathrm{Coh}(X) \to \mathrm{Coh}(X)$ is obviously t-exact, and by Lemma 1.3.1.7 so is $\mathcal{E} \otimes - : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X)$. $\square$

Corollary 1.3.3.7. The following version of projection formula holds.

$$f_*(\mathcal{E}_X) \otimes \mathcal{F}_Y \xrightarrow{\simeq} f_*^{\mathrm{IndCoh}}\left(\mathcal{E}_X \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_Y)\right)$$

Proof: It suffices to construct the natural transformation and prove that it is an equivalence on the level of functors $\mathrm{QCoh}(X)^{\omega} \otimes \mathrm{Coh}(Y) \to \mathrm{IndCoh}(Y)$, and both sides are ind-extended from here. Note that after post-composing Ψ_Y, this is just the ordinary projection formula for QCoh, which is an equivalence. Then, we have $f^{\mathrm{IndCoh},*}(\mathcal{F}_Y) = f^*(\mathcal{F}_Y) \in \mathrm{Coh}(X)$, so by Construction 1.3.1.11 we have $\mathcal{E}_X \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_Y) \in \mathrm{Coh}(X)$. Finally, by Construction 1.3.2.1 we have f_*^{IndCoh} preserves coherent sheaves, so we obtain

$$f_*^{\mathrm{IndCoh}}\left(\mathcal{E}_X \otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_Y)\right) \in \mathrm{Coh}(Y) \subset \mathrm{IndCoh}(Y)^+$$

Hence, it suffices to prove

$$f_*(\mathcal{E}_X) \otimes \mathcal{F}_Y \in \mathrm{IndCoh}\,(Y)^+$$

where Ψ_Y is an equivalence.

By the analogous steps of the proof in Lemma 1.3.2.9, we have $f_*(\mathcal{E}_X)$ finite tor amplitude. By Lemma 1.3.1.12, the desired conclusion follows. $\square$

Then, we investigate more properties of the action of QCoh on IndCoh. Namely, we aim to prove the following property.

Proposition 1.3.3.8. We have a fully faithful functor

$$\left(\mathrm{id}_{\mathrm{QCoh}(X)} \otimes f^{\mathrm{IndCoh},*}\right) : \mathrm{QCoh}(X) \otimes_{\mathrm{QCoh}(Y)} \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$$

Proof: The left hand side is compactly generated by objects of the form $\mathcal{E}_X \otimes \mathcal{F}_Y$, where $\mathcal{E}_X \in \mathrm{QCoh}(X)^{\omega}$ and $\mathcal{F}_Y \in \mathrm{Coh}(Y)$. Moreover, above functor preserves compact objects. We thus suffices to show for compact $\mathcal{E}_X^1, \mathcal{E}_X^2, \mathcal{F}_Y^1, \mathcal{F}_Y^2$, the following equivalence

$$\mathrm{Hom}_{\mathrm{QCoh}(X)\otimes_{\mathrm{QCoh}(Y)}\mathrm{IndCoh}(Y)}(\mathcal{E}_X^1\otimes\mathcal{F}_Y^1,\mathcal{E}_X^2\otimes\mathcal{F}_Y^2)\xrightarrow{\simeq}$$
$$\mathrm{Hom}_{\mathrm{IndCoh}(X)}\left(\mathcal{E}_X^1\otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_Y^1),\mathcal{E}_X^2\otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_Y^2)\right)$$

By dualizability of $\mathcal{E}_X^1$, by replacing $\mathcal{E}_X^2$ by tensoring it with $(\mathcal{E}_X^1)^\vee$, it suffices to show

$$\mathrm{Hom}_{\mathrm{QCoh}(X)\otimes_{\mathrm{QCoh}(Y)}\mathrm{IndCoh}(Y)}(\mathcal{F}_Y^1,\mathcal{E}_X\otimes\mathcal{F}_Y^2)\xrightarrow{\simeq}\mathrm{Hom}_{\mathrm{IndCoh}(X)}\left(f^{\mathrm{IndCoh},*}(\mathcal{F}_Y^1),\mathcal{E}_X\otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_Y^2)\right)$$

Since $f^{\mathrm{IndCoh},*}$ is left adjoint to f_*^{IndCoh}, it suffices to show that

$$\mathrm{Hom}_{\mathrm{QCoh}(X)\otimes_{\mathrm{QCoh}(Y)}\mathrm{IndCoh}(Y)}(\mathcal{F}_Y^1,\mathcal{E}_X\otimes\mathcal{F}_Y^2)\xrightarrow{\simeq}\mathrm{Hom}_{\mathrm{IndCoh}(Y)}\left(\mathcal{F}_Y^1,f_*^{\mathrm{IndCoh}}(\mathcal{E}_X\otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_Y^2))\right)$$

By Corollary 1.3.3.7, we have

$$f_*^{\mathrm{IndCoh}}\left(\mathcal{E}_X\otimes f^{\mathrm{IndCoh},*}(\mathcal{F}_Y^2)\right)=f_*(\mathcal{E}_X)\otimes\mathcal{F}_Y^2$$

and the rest is by definition. □

Corollary 1.3.3.9. The map in Proposition 1.3.3.8 is an equivalence when f is an open immersion.

Proof: The essential image of the functor clearly generates the target category. □

1.3.4. The exceptional pullback functor

Definition 1.3.4.1. Let $f: X\to Y\in\mathbf{Sch}_{\mathrm{aft}}$ be a functor. We say f is proper (or closed embedding) if f_{cl} is so.

Lemma 1.3.4.2. If $f: X\to Y$ is proper, then the functor $f_*^{\mathrm{IndCoh}}:\mathrm{IndCoh}(X)\to\mathrm{IndCoh}(Y)$ preserves compact objects. That is, it sends $\mathrm{Coh}(X)$ to $\mathrm{Coh}(Y)$.

Proof: By Construction 1.3.2.1, it suffices to show that $f_*:\mathrm{QCoh}(X)\to\mathrm{QCoh}(Y)$ sends $\mathrm{Coh}(X)$ to $\mathrm{Coh}(Y)$. By definition, for f a closed embedding this obviously holds. Then, it suffices to prove f_* maps $\mathrm{Coh}(X)^\heartsuit$ to $\mathrm{Coh}(Y)$. Then, noticing $\mathrm{Coh}(X)^\heartsuit=\mathrm{Coh}(X_{\mathrm{cl}})^\heartsuit$, we may assume X is classical. Then, the map $X\to Y$ factors through Y_{cl}, and since $Y_{\mathrm{cl}}\hookrightarrow Y$ is a closed embedding, we may assume that Y is also classical.

Thus, we are reduced to proving that for a proper map of classical schemes of finite type $f: X\to Y$ over a field (hence Noetherian), we have $R^if_*(\mathcal{F})$ coherent for any coherent $\mathcal{F}$. This is [Sta26] 02O5.

We will recall and use [Sta26] 01YI. Let X be a Noetherian scheme and $\mathcal{P}$ a property for coherent sheaves over X. Assume that $\mathcal{P}$ has the two-out-of-three property for any exact sequence, and for every integral closed subscheme $Z\subset X$ with generic point ξ, there is a coherent sheaf $\mathcal{G}\in\mathrm{Coh}(X)$ satisfying the following properties:
(1) it is supported on Z,
(2) $\mathcal{G}_\xi$ is annihilated by $\mathfrak{m}_\xi$,
(3) $\dim_{\kappa(\xi)}\mathcal{G}_\xi=1$
(4) $\mathcal{G}$ has property $\mathcal{P}$

then $\mathcal{P}$ holds for any coherent sheaf over X. The proof of this lemma is basically a version of Noetherian induction.

Now, we set $\mathcal{P}$ to be $R^i f_*(\mathcal{F})$ is coherent for all i. We verify the properties above. Firstly, $\mathcal{P}$ satisfy the two-out-of-three for exact sequences is obvious by the long exact sequence of cohomology. Then, take $Z \subset X$ an integral closed subscheme. We wish to find a coherent sheaf satisfying the above properties. We let $g : Z \to Y$ denote the restriction of f. We will actually construct a coherent sheaf $\mathcal{G}$ over Z such that $g_*(\mathcal{G})$ has no higher images, $\mathcal{G}_\xi$ is 1-dimensional, and $g_*(\mathcal{C})$ is coherent. Then, pushing $\mathcal{G}$ to X gives a coherent sheaf with the desired properties. Thus everything reduces to finding such a coherent sheaf $\mathcal{G}$.

We apply Chow's lemma to $Z \to Y$. Namely, there is a proper Y-scheme map $\pi : Z' \twoheadrightarrow Z$, with a dense open $U \subset Z$ such that $\pi^{-1}(U) \to U$ is an isomorphism, and $i : Z' \hookrightarrow \mathbb{P}^n_Y$. By properness of π, the diagonal map $i' : Z' \hookrightarrow \mathbb{P}^n_Y \times_Y Z = \mathbb{P}^n_Z$ is a closed immersion. We take a line bundle

$$\mathcal{L} = i^*\Big(\mathcal{O}_{\mathbb{P}^n_Y}(1)\Big) = i'^*\Big(\mathcal{O}_{\mathbb{P}^n_Z}(1)\Big)$$

which is ample for both $g' : Z' \to S$ and $\pi : Z' \to Z$. By the equivalent characterization of ampleness, there is some m such that the higher direct images of $\mathcal{L}^{\otimes m}$ with respect to g' and π vanishes. We then set $\mathcal{G} = \pi_*(\mathcal{L}^{\otimes n})$. Note $\mathcal{G}$ is invertible over U since π is an isomorphism over U, we have $\mathcal{G}_\xi$ dimension 1 over $\kappa(\xi)$. The higher direct images vanishes by a computation of the Leray-Serre spectral sequence

$$E_2^{p,q} = R^p g_* \circ R^q \pi_*(\mathcal{L}^{\otimes n}) \Longrightarrow R^{p+q} g'_*(\mathcal{L}^{\otimes n})$$

we have the left hand side is non-zero iff $q = 0$, and the right hand side is non-zero iff $p + q = 0$, so $R^p g_*(\mathcal{G}) = 0$ for $p > 0$. Then, since g' is projective, $g_*(\mathcal{G}) = g'_*(\mathcal{L}^{\otimes n})$ is clearly coherent by a computation of the cohomology of line bundles over projective spaces. $\square$

1.4. Application: 1-Cartier duality

The contents in this section may be treated over the more general base **Sp**. That is, we no longer suppose the base ring we are considering over is any field of characteristic 0, but over the sphere spectrum $\mathbb{S}$. The readers may assume that all the statements needed still holds.

2. Todo chapter

todo 2.1.

3. The geometric Satake equivalence

3.1. The affine Grassmannian

3.1.1. Basic constructions

Notation 3.1.1.1. Let k be a field, and $F = k((t))$ the local function field. Then let $\mathcal{O} = k[[t]]$ denote the ring of integers. We let $D_R = \operatorname{Spec} R[[t]]$ and $D_R^* = \operatorname{Spec} R((t))$ denote the (punctured) formal disc.

Definition 3.1.1.1. Let G be a smooth affine k-group. We let Gr_G, the affine Grassmannian of G be the functor of points given by $\mathrm{Gr}_{G(R)} = \{(\mathcal{E}, \beta)\}$, where $\mathcal{E}$ is a G-torsor on D_R and $\beta : \mathcal{E}|_{D_R^*} \to \mathcal{E}^0|_{D_R^*}$ be a trivialization of $\mathcal{E}$ on the punctured disc (here $\mathcal{E}^0$ is the trivial bundle). We shall call the data β a modification of the bundle $\mathcal{E}$, sometimes denoted by $\beta : \mathcal{E} \dashrightarrow \mathcal{E}^0$

Note that this definition makes sense for general group schemes $\underline{G}$ over $\mathcal{O}$ or D_k (whereas in the above case $\underline{G} = G_{\mathcal{O}}$), which we also denote by $\mathrm{Gr}_{\underline{G}}$.

Example 3.1.1.2. As a first example, take $G = \mathrm{GL}_n$. Then $\mathrm{Gr}_{G(R)}$ admits another functor of points description, given by the set of lattices $\Lambda \hookrightarrow R((t))^n$, where Λ is a projective $R[[t]]$-module such that $\Lambda \otimes_{R[[t]]} R((t)) = R((t))^n$. To see that this is equivalent to the above definition, notice that a GL_n-principal bundle is equivalent to an n-dimensional vector bundle, and the projective module is the data for $\mathcal{E}$, and the second isomorphism is the trivialization away from 0.

Proposition 3.1.1.2. $\mathrm{Gr}_{\underline{G}}$ is represented by an ind-scheme ind-of-finite-type over k. Moreover, if $\underline{G}$ is reductive, then $\mathrm{Gr}_{\underline{G}}$ is ind-projective.

Proof: We only prove the case where G is constant, since the general fact relies on the fact that for any group $\underline{G}$ over $k[[t]]$, there is a faithful representation $\underline{G} \hookrightarrow \mathrm{GL}_n$ for some n such that $\frac{\mathrm{GL}_n}{\underline{G}}$ is quasi-affine, and when $\underline{G}$ is reductive, it is affine. In the constant case, we can obviously embed G into GL_n over k with the above properties and base change to $k[[t]]$. In this case, we have a locally closed embedding $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$ which is closed in the reductive case. It thus remains to show that $\mathrm{Gr}_{\mathrm{GL}_n}$ is ind-projective.

To show this, we note that for every lattice $\Lambda \hookrightarrow R((t))^n$, there is some N such that $t^{-N}R[[t]]^n \hookrightarrow \Lambda \hookrightarrow t^N R[[t]]^n$. By definition, the functor of points for lattices Λ between $t^{-N}R[[t]]^n$ and $t^N R[[t]]^n$ is $\mathrm{Gr}(2nN)$, the usual grassmannian. By the Plucker embedding, $\mathrm{Gr}(2nN)$ is projective, and by definition $\mathrm{Gr}_{\mathrm{GL}_n} = \operatorname{colim}_N \mathrm{Gr}(2nN)$. Moreover, note that the transition maps are closed embeddings, so we have $\mathrm{Gr}_{\mathrm{GL}_n}$ ind-projective. $\square$

We then give another characterization of the affine Grassmannian.

Definition 3.1.1.3. Let X denote a presheaf over $\mathcal{O}$. We define the n-th jet space of X L^nX to be the presheaf over k such that $L^nX(R) = X\left(\frac{R[t]}{t^n}\right)$. Moreover, we define the positive loop space of X L^+X to be the presheaf over k such that $L^+X(R) = X(R[[t]])$.

Then let X denote a presheaf over K. We define the loop space LX to be the presheaf over k such that $LX(R) = X(R((t)))$.

For X a presheaf over k, we write L^nX, L^+X, LX short for $L^n(X \otimes_k \mathcal{O})$, $L^+(X \otimes_k \mathcal{O})$, and $L(X \otimes_k K)$.

Example 3.1.1.3. In particular, for $\underline{G}$ a group scheme over $\mathcal{O}$, we may define the loop group LG and the positive loop group L^+G, which would be of fundamental importance in the theory.

Theorem 3.1.1.4. $\mathrm{Gr}_{\underline{G}} = LG/L^+G$, where the right hand side is identified with the fpqc quotient stack.

Proof: We first construct a natural action $L\underline{G} \circlearrowleft \mathrm{Gr}_{\underline{G}}$, given on R-points by $g \cdot (\mathcal{E}, \beta) = (\mathcal{E}, g\beta)$. Notice that we have a natural identification of $L\underline{G}(R)$ with the set of data $(\mathcal{E}, \beta, \varepsilon)$, where $(\mathcal{E}, \beta) \in \mathrm{Gr}_{\underline{G}}(R)$, and $\varepsilon : \mathcal{E} \to \mathcal{E}^0$ is a global trivialization. We have this isomorphism by mapping $g \in L\underline{G}(R)$ to $g \mapsto (\mathcal{E}^0, g, \mathrm{id})$ and the inverse $(\mathcal{E}, \beta, \varepsilon) \mapsto \beta \cdot \varepsilon$. Thus the projection $L\underline{G} \twoheadrightarrow \mathrm{Gr}_{\underline{G}}$ to the first two factors clearly identifies $L\underline{G}$ as a $L^+\underline{G}$-torsor over $\mathrm{Gr}_{\underline{G}}$. □

We now introduce the global version of the affine Grassmannian.

Definition 3.1.1.5. Let X be a reduced connected curve over k (not necessarily smooth), and let $x \in |X|$ be a closed point. Let $\underline{G}$ be a smooth affine group scheme over X, and we define $\mathrm{Gr}_{\underline{G},x}$ to be the presheaf over k sending R to pairs $(\mathcal{E}, \beta)$, where $\mathcal{E}$ is a $\underline{G}$-torsor on X_R, and $\beta : \mathcal{E}|_{X_R^*} \to \mathcal{E}^0|_{X_R^*}$ be a trivialization of $\mathcal{E}$ away from x (here X_R^* is the the base change of $X^* = X - x$ to R).

Definition 3.1.1.6. Let $x \in |X|$ be a closed point in a reduced connected curve over k. We let $\mathcal{O}_x$ denote the completed local ring at x (since we will not use the usual stalk in this section), and let $D_{x,R}$ denote $\operatorname{Spec} \mathcal{O}_x \hat{\otimes} R$.

Definition 3.1.1.7. Let $\underline{G}_x = \underline{G} \times_X x$. Moreover, let $\mathrm{res} : \mathrm{Gr}_{\underline{G},x} \to \mathrm{Gr}_{\underline{G}_x}$ induced by restricting a $\underline{G}$ -torsor on X_R to $D_{x,R}$.

Proposition 3.1.1.8. The above restriction map is an isomorphism.

To prove Proposition 3.1.1.8, we need the Beauville-Laszlo's descent theorem.

Theorem 3.1.1.9 (Beauville-Laszlo). Let $p : A \to \tilde{A}$ be a map of commutative rings. We fix a non-zero-divisor $f \in A$ and assume that $p(f)$ is also not a zero divisor. Moreover, we assume that the natural map $A/f \to \tilde{A}/p(f)$ is an isomorphism. Let $\mathbf{Mod}_A^f$ denote the f-torsion-free A-modules, and

$\mathrm{Desc}\big(A, \tilde{A}, f\big)$ denote the category of the descent datum (M_1, M_2, φ) where $M_1 \in \mathbf{Mod}_{A[f^{-1}]}$, $M_2 \in \mathbf{Mod}_{\tilde{A}}^{p(f)}$, and $\varphi : M_1 \otimes_{A[f^{-1}]} \tilde{A}[p(f)^{-1}] \to M_2 \otimes_{\tilde{A}} \tilde{A}[p(f)^{-1}]$ is an isomorphism.

Now, by flatness of localization, we have $\varphi : M[f^{-1}] \otimes_{A[f^{-1}]} \tilde{A}[p(f)^{-1}] \simeq \big(M \otimes_A \tilde{A}\big)[p(f)^{-1}]$, so $M \mapsto \big(M[f^{-1}], M \otimes_A \tilde{A}, \varphi\big)$ gives a functor $\mathbf{Mod}_A^f \to \mathrm{Desc}\big(A, \tilde{A}, f\big)$. This functor is then an equivalence, and finiteness, flatness, and finite projectivity are preserved under descent.

Proof: We now prove Proposition 3.1.1.8. We directly construct an inverse to res, namely, given an $\underline{G}_x$-torsor on $D_{x,R}$ with a trivialization β on $D^*_{x,R}$, we glue it with the trivial $\underline{G}$-torsor on X^*_R by β via Theorem 3.1.1.9 (we should find a vector bundle on X such that it is the associated bundle of our principal bundles, but I'm temporarily not quite sure on how to do this). □

Corollary 3.1.1.10. The loop group $L\underline{G}_x$ represents the k-presheaf sending R to triples $(\mathcal{E}, \alpha, \beta)$, $\mathcal{E}$ is a $\underline{G}$-torsor on X, α a trivialization of $\mathcal{E}|_{D_{x,R}}$, and β a trivialization of $\mathcal{E}|_{X^*_R}$.

remark 3.1.1.4. There is an alternate construction of loopgroups and affine Grassmannians over the mixed characteristic world, sometimes called the Witt Grassmannian. Namely, this time, $L^+G(R) = G(W(R))$, $LG(R) = G(W(R)[p^{-1}])$, $\mathrm{Gr}_G = LG/L^+G$. They are sheaves over the perfect algebras over $\mathbb{F}_p$.

3.1.2. The affine Weyl group

Notation 3.1.2.1. From now on, we assume G is a reductive group over k. We fix a Borel subgroup $B \subset G$, and its maximal torus T. Denote the weight (character) lattice and coweight (cocharacter) lattice by $X^\bullet(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ and $X_\bullet(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ respectively. We let $\Phi \subset X^\bullet(T)$ denote the set of roots, and $\Phi^\vee \subset X_\bullet(T)$ denote the coroots. We denote the positive roots by Φ^+, and the dominant coweights by $X_\bullet(T)^+$.

For $\mu \in X_\bullet(T)$ a coweight, we denote t^μ by the image of $t \in \mathbb{G}_m(K)$ by μ in $T(K)$.

Theorem 3.1.2.1. We have an natural isomorphism $G(\mathcal{O})\backslash G(F)/G(\mathcal{O}) \simeq X_\bullet(T)/W = X_\bullet(T)^+$.

Proof: Recall that we have the Cartan decomposition by linear algebra

$$G(F) = \coprod_{\mu \in X_{\bullet(T)}} G(\mathcal{O}) t^\mu G(\mathcal{O})$$

which gives the desired decomposition. □

Construction 3.1.2.2. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two trivial G-torsors over D with trivializations $\varphi_i : \mathcal{E}_i \to \mathcal{E}^0$, and let $\beta : \mathcal{E}_1|_{D^*} \to \mathcal{E}_2|_{D^*}$ be an isomorphism over D^*. Then, the composition $\varphi_2 \beta \varphi_1^{-1} \in \mathrm{Aut}(\mathcal{E}^0|_{D^*}) = G(F)$, so obviously its image in $G(\mathcal{O})\backslash G(F)/G(\mathcal{O}) = X_\bullet(T)^+$ does not depend on the trivializations φ_i, denoted by $\mathrm{Inv}(\beta)$, the relative position of the modification β.

Now, given the Bruhat decomposition, as in the classical case, we may define the Schubert varieties. The Schubert varieties defined via the dominant coweights by the spherical Schubert varieties. But before we start, we need a lemma on group actions and orbits.

Lemma 3.1.2.3 (orbit lemma). Let G be a smooth group scheme and X an algebraic variety. Let $G \circlearrowright X$, then for any $x \in X$, the orbit $G \cdot x \subset X$ is locally closed in X.

We omit the proof, but the key of the proof is using that the image of maps of varieties is constructible, and finally using a dense open set to deduce locally closedness.

Definition 3.1.2.4. Recall $L^+G \circlearrowright LG$ hence on the quotient LG/L^+G (note that we take quotient on the right here, so the action is not trivial). We let Gr_μ denote the L^+G-orbit (naturally equipped with the reduced structure) of the image of the point $t^\mu \in LG \twoheadrightarrow \mathrm{Gr}$. Then, we let

$$\mathrm{Gr}_{\leqslant\mu} = \coprod_{\lambda\leqslant\mu} \mathrm{Gr}_\lambda$$

We call $\mathrm{Gr}_{\leqslant\mu}$ the spherical Schubert varieties and Gr_μ the spherical Schubert cell.

Alternatively, one may directly describe the locally closed subset Gr_μ via the modification.

Proposition 3.1.2.5. Gr_μ can also be identified as the subset of Gr

$$\mathrm{Gr}_\mu = \{(\mathcal{E}, \beta) \in \mathrm{Gr} |\, \mathrm{Inv}(\beta) = \mu\}$$

Proof: Firstly, note that $t^\mu = (\mathcal{E}, \beta) \in \mathrm{Gr}$ has obviously $\mathrm{Inv}(\beta) = \mu$ by the definition of relative position and the projection $LG \to \mathrm{Gr}$. Then, note that for $(\mathcal{E}', \beta') = g(\mathcal{E}, \beta)$ for some $g \in L^+G$, we have $\mathrm{Inv}(\beta') = \mathrm{Inv}(\beta)$ since the action of g on $(\mathcal{E}, \beta)$ is given by left multiplication on β, and in the definition of relative position, we have a quotient of $G(\mathcal{O})$ on the left. Thus, in a single L^+G-orbit, the relative position of β is always the same. Finally, by Theorem 3.1.2.1, we have

$$\mathrm{Gr} = \coprod (L^+G \backslash LG/L^+G)/L^+G = \coprod \mathrm{Gr}_\mu$$

so each Gr_μ is exactly the orbit such that the invariant is equal to μ. □

remark 3.1.2.2. There is a slight subtlety in the above proof by confusing the set $G(F)$ $(G(\mathcal{O}))$ with the ind-scheme LG (L^+G). However, since everything is reduced, and we are working over an algebraically closed field, no serious problem arises since any closed reduced subscheme over a algebraically closed fields are determined by its closed points.

Proposition 3.1.2.6. Gr_μ is a quasi-projective variety of dimension $\langle 2\rho, \mu\rangle$. Moreover, we have $\mathrm{Gr}_{\leqslant\mu}$ the Zariski closure of Gr_μ, so is a (not necessarily smooth) projective variety.

Proof: By definition, the stabilizer for $L^+G \circlearrowright \mathrm{Gr}$ of t^μ is $L^+G/(L^+G \cap t^\mu L^+G t^{-\mu})$, which is firstly finite dimensional by calculating the dimension of its tangent space, which is given by

$$\mathfrak{g}(\mathcal{O})/(\mathfrak{g}(\mathcal{O}) \cap \mathrm{Ad}_{t^\mu} \mathfrak{g}(\mathcal{O})) = \bigoplus_{\langle\alpha,\mu\rangle \geqslant 0} \mathfrak{g}_\alpha(\mathcal{O})/t^{\langle\alpha,\mu\rangle}\mathfrak{g}_\alpha$$

so $\dim \mathrm{Gr}_\mu = \langle 2\rho, \mu\rangle$. Then, since it is finite dimensional locally closed subscheme of an ind-projective variety Gr, it is quasi-projective.

The second assertion is more difficult. We show that the Zariski closure of Gr_μ contains Gr_λ for any $\lambda \leqslant \mu$, and assume the spherical Schubert variety $\mathrm{Gr}_{\leqslant\mu}$ is closed in Gr, so identifying $\mathrm{Gr}_{\leqslant\mu}$ as the closure of Gr_μ. Then, recall that for $\lambda < \mu$, there is a sequence μ_i such that $\lambda = \mu_0 < \mu_1 < ... < \mu_n = \mu$, where μ_i and μ_{i-1} differ by a positive simple root. Thus, it suffices to show that $t^{\mu-\alpha}$ lies in the closure of Gr_μ. To show this, we construct a curve C connecting the two points $t^{\mu-\alpha}$ □

Lemma 3.1.2.7. Θ is an effective Cartier divisor corresponding to the $\mathcal{O}(1)$-bundle.

3.2. The proof of the Satake equivalence

3.2.1. The Satake category

3.2.2. The geometric Satake transform

Now that we have proven that the functor of taking hypercohomology $H^* : \mathrm{Sat}_G \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$ is a neutral Tannakian category, we may identify Sat_G with the representation category $\mathbf{Rep}_{\tilde{G}}$, where the Tannakian group $\tilde{G}$ is the algebraic group of automorphisms $\mathrm{Aut}^\otimes H^*$. We now (and the proof of theorem suffices to) identify $\tilde{G}$ with the dual group of G over $\overline{\mathbb{Q}}_\ell$.

Definition 3.2.2.1. Recall a reductive group is completely determined by its root datum $(X_\bullet(T), \Phi^\vee, X^\bullet(T), \Phi)$. For a reductive group G over a algebraically closed field with its root datum given by $(X_\bullet(T), \Phi^\vee, X^\bullet(T), \Phi)$, its dual group $\hat{G}$ to be the reductive group with root datum $(X^\bullet(T), \Phi, X_\bullet(T), \Phi^\vee)$.

Notice that the base field of $\hat{G}$ can be arbitrary.

Example 3.2.2.1. GL_n has dual GL_n, and SL_n has dual PGL_n.

Lemma 3.2.2.2. $\tilde{G}$ is a connected reductive group over $\overline{\mathbb{Q}}_\ell$.

Proof: Firstly, we claim that the sheaves IC_μ for μ the fundamental coweights (the weights dual to the simple roots) form a set of tensor generators for Sat_G. Notice that the convolution Grassmannian $\mathrm{Gr}_{\leqslant\mu} \tilde{\times} \mathrm{Gr}_{\leqslant\nu} \to \mathrm{Gr}_{\leqslant\mu+\nu}$ is birational. Now, given a proper birational map $f : X \to Y$, we prove IC_Y is a direct summand of $f_*\mathrm{IC}_X$. Now, take an open set U contained in the smooth locus of Y such that f is an isomorphism on U. By definition, $f_*\mathrm{IC}_X|_U = \mathrm{IC}_Y|_U = k_U$. By the perverse continuation principle applied to the sheaves $f_*\mathrm{IC}_X|_U$ and $\mathrm{IC}_Y|_U$, we have a map $\mathrm{IC}_Y \to f_*\mathrm{IC}_X$ non-zero on U, so is non-zero. Thus since IC_Y is simple, it must be a subobject of $f_*\mathrm{IC}_X$. Thus applied to our previous settings, $\mathrm{IC}_{\mu+\nu}$ is a subobject hence a direct summand of the convolution sheaf $\mathrm{IC}_\mu * \mathrm{IC}_\nu$ by the decomposition theorem.

This provides that $\tilde{G}$ is connected and algebraic (i.e. the algebra $\mathcal{O}_{\tilde{G}}\left(\tilde{G}\right)$ is of finite type) because $\bigotimes_\mu \mathrm{IC}_\mu$ for μ running over fundamental coweights is a tensor generator for Sat_G. Then, since Sat_G is semisimple, we have furthermore that $\tilde{G}$ is reductive. □

Theorem 3.2.2.3. Letting $(X^\bullet(T), \Phi^\vee, X_\bullet(T), \Phi)$ denote the root datum of G, we have the Tannakian group $\tilde{G}$ has root datum $(X_\bullet(T), \Phi, X^\bullet(T), \Phi^\vee)$, so $\tilde{G}$ can be identified with the dual group of G.

The proof of Theorem 3.2.2.3 is by first proving the easy case where G is a torus, then reducing general cases to tori.

Now, in general, the affine Grassmannians can be very non-reduced, but since we are talking about ℓ-adic sheaves over the spaces, we may temporarily get rid ourselves of the nuisances by taking $(\mathrm{Gr}_G)_{\mathrm{red}}$. Thus, from now on, Gr_G will naturally mean the reduced versions without further notices.

Lemma 3.2.2.4. Theorem 3.2.2.3 holds if $G = T$ is a torus.

Proof: We first discuss the structure of Gr_T. By definition, let $T = \mathbb{G}_m^n$, then $LT(R) = T(R((t))) = (R((t))^\times)^n$, and $L^+T(R) = (R[[t]]^\times)^n$. Then, by definition, we have $R((t))^\times / R[[t]]^\times$ identified by $\mathbb{Z}$, identified to the lowest exponent of t in each component. Thus this tuple of integers is given equivalently by a cocharacter $\lambda \in X_\bullet(T) \simeq \mathbb{Z}^n$. Therefore, Gr_T is a discrete set with points identified with $X_\bullet(T)$. Thus, Sat_T is the category of $X_\bullet(T)$-graded finite dimensional vector spaces, and H^* is the functor forgetting the grading.

Now, since $X_\bullet(T)$-graded vector spaces is tensor generated by 1_μ where μ are fundamental coweights, so an R-point of the Tannakian groups is given by maps $1_\mu \to R^\times$, hence by the tensor-automorphism condition, are given by $\mathrm{Hom}_{\mathbf{Ab}}(X_\bullet(T), R^\times)$. And by definition, this is the functor of points of the dual torus. □

Now it suffices to reduce the case to $G = T$. To do this, we construct the categorical Satake transform

$$\mathrm{CT} : \mathrm{Sat}_G \to \mathrm{Sat}_T$$

which is a symmetric monoidal functor from the Satake category of G to the one of its maximal torus T, which firstly fits in the following commutative diagram

$$\begin{tikzcd} \mathrm{Sat}_G \arrow[rr, "\mathrm{CT}"] \arrow[dr, "H^*"'] & & \mathrm{Sat}_T \arrow[dl, "H^*"] \\ & \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell} & \end{tikzcd}$$

Construction 3.2.2.5. The main idea of constructing CT is via parabolic induction. We fix a borel subgroup $i : B \subset G$ and let $q : B \to T$ denote the projection to the abstract Cartan subgroup. The maps induces maps on the affine Grassmannians

$$\mathrm{Gr}_T \xleftarrow{q} \mathrm{Gr}_B \xrightarrow{i} \mathrm{Gr}_G$$

Then, we have $\mathrm{Gr}_T \simeq X_\bullet(T)$, and for $\lambda \in X_\bullet(T)$, we denote $S_\lambda = i(q^{-1}(\lambda)) \subset \mathrm{Gr}_G$, which, by definition, can also be identified with $LU \cdot t^\lambda$, where $U \subset B$ is the unipotent subgroup. Moreover, we define

$$S_{\leqslant\lambda} = \bigcup_{\lambda' \leqslant \lambda} S_{\lambda'}$$

To construct and investigate the properties of the parabolic inductions, we start with some topological (and finally, cohomological) properties of S_λ.

Proposition 3.2.2.6. $S_{\leqslant\lambda}$ is closed with $S_\lambda \subset S_{\leqslant\lambda}$ an open dense subset.

Proof: We first show that $S_{\leqslant\lambda}$ is closed. We consider its moduli description. Let χ be a dominant integral weight, and V_χ denote the highest weight representation of G, and ℓ_χ denote its highest weight space. Then we have

$$S_{\leqslant\lambda} = \left\{(\mathcal{E}, \beta) \in \mathrm{Gr} | \beta^{-1}\left(\ell_\chi\right) \subset t^{-(\chi,\lambda)}\left(V_{\chi,\mathcal{E}}\right)\right\}$$

where $V_{\chi,\mathcal{E}}$ is the associated bundle of $\mathcal{E}$, namely, $\mathcal{E} \times^G V_\chi$, which is a vector bundle over D_k. Then, these conditions can be translated to a polynomial equation by passing to matrices, so is a closed subscheme of Gr_G.

Then, for the density of $S_\lambda \subset S_{\leqslant\lambda}$, it actually follows from the construction of the curve in Proposition 3.1.2.6. □

Lemma 3.2.2.7. The set $S_{<\lambda}$ is the intersection of $\overline{S}_\lambda$ with a hyperplane section of the bundle $\mathcal{O}(1)$.

Proof: Let $2\rho^\vee$ denote the sum of positive roots of G. Then we choose some embedding $T \subset B$, and consider the map

$$\mathbb{G}_m \xrightarrow{2\rho^\vee} T \subset G \subset L^+G$$

Then the L^+G action on Gr_G restricts to a $\mathbb{G}_m$-action on Gr_G via the above inclusion. Notice that the fixed points of this action is given by Gr_T by the Cartan decomposition, and we can identify the attractor of this action with Gr_B. □

Theorem 3.2.2.8. We have an isomorphism of functors

$$H^*(-) = \bigoplus_\lambda H_c^*(S_\lambda, -) : \mathrm{Sat}_G \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$$

Proof:

□

4. The Bezrukavnikov-Finkelberg equivalence

We attempt to give a rather complete sketch of the infamous (at least among my friends) article of Bezrukavnikov-Finkelberg. The main theorem of the article is the following:

Theorem 4.1. $D^{\check{G}}_{\mathrm{perf}}(U^{\square}_{\hbar}) \simeq D_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr})$, and $D^{\check{G}}_{\mathrm{perf}}(\mathrm{Sym}^{\square}(\check{\mathfrak{g}})) \simeq D_{L^+G}(\mathrm{Gr})$.

where the algebras $U^{\square}_{\hbar}$ and $\mathrm{Sym}^{\square}(\check{\mathfrak{g}})$ are to be defined later. The strategy of the proof is given by the following steps.

(1) Construct functors $\kappa_\hbar : \mathcal{HC}_\hbar \to \mathrm{QCoh}^{\mathbb{G}_m}\left(M_{(\check{\mathfrak{t}}/W)^2}\Delta\right)$ and $\kappa : \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}}^*) \to \mathrm{Coh}^{\mathbb{G}_m}\left(T\left(\check{\mathfrak{t}}^*/W\right)\right)$. The functors are fully faithful on the subcategory of free objects, namely $\mathcal{HC}^{\mathrm{fr}}_\hbar$ and $\mathrm{Coh}^{\check{G}\times\mathbb{G}_M}_{\mathrm{fr}}(\check{\mathfrak{g}}^*)$.

(2) Extend the Satake functor $S : \mathbf{Rep}\left(\check{G}\right) \to \mathbf{Perv}_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr})$ to fully faithful functors

$$S_\hbar : \mathcal{HC}^{\mathrm{fr}}_\hbar \to D_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr}), \text{and } S_{qc} : \mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*) \to D_{L^+G}(\mathrm{Gr})$$

These functors are constructed via the fully faithful cohomology functors

$$H^*_{L^+G\rtimes\mathbb{G}_m} : \mathbf{Perv}_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr}) \to \mathrm{Coh}^{\mathbb{G}_m}\left(M_{(\check{\mathfrak{t}}/w)^2}\Delta\right), H^*_{L^+G} : \mathbf{Perv}_{L^+G}(\mathrm{Gr}) \to \mathrm{Coh}^{\mathbb{G}_m}\left(T\left(\check{\mathfrak{t}}^*/W\right)\right)$$

and by analysing the images, we have $\kappa_\hbar|_{\mathcal{HC}^{\mathrm{fr}}_\hbar}$ and $\kappa|_{\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*)}$ factors through the cohomology functors by $S_\hbar$ and S_{qc}, i.e., $\kappa_\hbar = H^*_{L^+G\rtimes\mathbb{G}_m} \circ S_\hbar$ and $\kappa = H^*_{L^+G} \circ S_{qc}$.

(3) Extend $S_\hbar$ and S_{qc} to our final equivalences via the fully faithful inclusions $\mathcal{HC}^{\mathrm{fr}}_\hbar \subset D^{\check{G}}_{\mathrm{perf}}(U^{\square}_\hbar)$ and $\mathrm{Coh}^{\check{G}\times\mathbb{G}_m}_{\mathrm{fr}}(\check{\mathfrak{g}}^*) \subset D^{\check{G}}_{\mathrm{perf}}(\mathrm{Sym}^{\square}(\check{\mathfrak{g}}))$.

We will expand the three steps one by one.

4.1. Step 1

4.1.1. Computations of equivariant cohomology

Before we start, we must compute the equivariant cohomologies of points.

Definition 4.1.1.1. Let X be a scheme and $Z \subset X$ be a closed subscheme classified by the quasi-coherent sheaf of ideals $\mathcal{I}$. We construct the deformation to the normal cone as following. Firstly, take $\mathbb{A}^1 = \mathrm{Spec}\, k[\hbar]$. Then, consider the following extended Rees $\mathcal{O}_X$-algebra $\mathcal{R}$ given by the sheaf of subalgebras in $\mathcal{O}_X[\hbar^\pm]$ generated by $\hbar$ and $f\hbar^{-1}$ where $f \in \mathcal{I}$. We define the deformation to the normal cone $M_X Z = \mathrm{Spec}_X\, \mathcal{R}$.

Definition 4.1.1.2. For groups G with $H^*_G(\mathrm{pt})$ is an integral domain, we define the localized equivariant cohomology $H^*_G(X)_{\mathrm{loc}} = H^*_G(X) \otimes_{H^*_G(\mathrm{pt})} \mathrm{Frac}(H^*_G(\mathrm{pt}))$.

Theorem 4.1.1.3. There is an equivalence

$$\alpha : \mathcal{O}\left(N_{(\check{\mathfrak{t}}/W)^2}\Delta\right) \simeq \bigoplus_{\pi_1(G)} H^*_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr})$$

Here Δ denote the diagonal embedding.

Proof: We first treat the case where G is simply connected.

Firstly, notice $H^*_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr}) = H^*_{\mathbb{G}_m}(L^+G\backslash LG/L^+G)$ admit a bimodule structure of $H^*_{L^+G}(\mathrm{pt})$. Since we have the homotopy equivalence $L^+G \simeq G$ by contractibility of $k[[t]]$, we have $H^*_{L^+G}(\mathrm{pt}) = H^*_G(\mathrm{pt}) = \mathcal{O}(\mathfrak{t}/W)$ by Chern-Weil theory. Thus, we have $H^*_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr})$ is a $H^*_{L^+G}(\mathrm{pt}) \otimes H^*_{L^+G}(\mathrm{pt}) \otimes H^*_{\mathbb{G}_m}(\mathrm{pt}) = \mathcal{O}((\mathfrak{t}/W)^2 \times \mathbb{A}^1)$. This gives a structure map

$$\alpha : \mathcal{O}((\mathfrak{t}/W)^2 \times \mathbb{A}^1) \to H^*_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr})$$

Then we show it extends to the extended Rees algebra $\mathcal{R}$, that is, we show that for $f \in \mathcal{I}_\Delta$, we have $\alpha(f)$ contains a factor of $\alpha(\hbar)$ where we regard $H^*_{\mathbb{G}_m}(\mathrm{pt}) = k[\hbar]$, i.e., the two maps $H^*_{L^+G}(\mathrm{pt}) \to H^*_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr})/\hbar$ coincide. This is because $H^*_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr})/\hbar = H^*(L^+G\backslash LG/L^+G)$. Thus we obtain a map

$$\alpha : \mathcal{O}\Big(M_{(\mathfrak{t}/W)^2}\Delta\Big) \to H^*_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr})$$

We then show it is an equivalence. For injectivity, note $H^*_{L^+G\rtimes\mathbb{G}_m}(\mathrm{pt}) = \mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)$. Notice by definition that $\mathcal{O}\Big(M_{(\mathfrak{t}/W)^2}\Delta\Big)$ is an integral domain, so it suffices to prove the localized morphism

$$\alpha_{\mathrm{loc}} : \mathcal{O}\Big(M_{(\mathfrak{t}/W)^2}\Delta\Big) \otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)} \mathrm{Frac}(\mathcal{O}(\mathfrak{t}\times\mathbb{A}^1)) \to H^*_{L^+G\rtimes\mathbb{G}_m}(\mathrm{Gr}) \otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)} \mathrm{Frac}(\mathcal{O}(\mathfrak{t}\times\mathbb{A}^1))$$

is an injection. This is because the right hand side can be identified with the localized equivariant cohomology $H^*_{T\times\mathbb{G}_m}(\mathrm{Gr})_{\mathrm{loc}}$. □

4.1.2. The involved categories

Definition 4.1.2.1. Let $U = U(\check{\mathfrak{g}})$, and let $U_\hbar$ denote the graded enveloping algebra, explicitly, the graded $k[\hbar]$-algebra generated by $\check{\mathfrak{g}}$ under the relations $xy - yx = \hbar[x, y]$. Equivalently, $U_\hbar$ is the Rees algebra of the filtered algebra U.

These enveloping algebras are equipped with the structure of $\check{G}$-representations via the adjoint action $\check{G} \circlearrowright \check{\mathfrak{g}}$.

Definition 4.1.2.2. We define the category of $\hbar$-Harish-Chandra bimodules $\widetilde{\mathcal{HC}}_\hbar$ to be the category with the following objects with the requirements:

(1) An object M is a graded $U_\hbar \otimes_{k[\hbar]} U_\hbar$-module equipped with the structure of a $\check{G}$-representation $\rho : \check{G} \circlearrowright M$, such that:
(2) the action map $U_\hbar \otimes_{k[\hbar]} U_\hbar \otimes M \to M$ is $\check{G}$-equivariant, and,
(3) for $x \in \check{\mathfrak{g}}$, the action of $1 \otimes x + x \otimes 1$ on M is $\hbar \cdot d\rho(x)$.

The intuition to this definition is some sort of Harish-Chandra modules of Harish-Chandra pairs.

We define $\mathcal{HC}_\hbar \subset \widetilde{\mathcal{HC}}_\hbar$ to be the full subcategory spanned by the modules M such that M is finitely generated as $U_\hbar \otimes 1$-modules.

5. Koszul duality

5.1. A first glance: Koszul duality for finite type groups

5.1.1. Highest weight categories

In this section, we introduce a special type of Abelian categories which behaves like the category of perverse sheaves over a suitable stratified space or the BGG category $\mathcal{O}$. The history of the developments of such categories is complicated, resulting in various definitions of such categories (some of which are simple generalizations of others). We will begin with the first definition.

Definition 5.1.1.1. Let $\mathcal{A}$ be an Abelian category over some field k. We say $\mathcal{A}$ is highest weight in the sense of [CPS88] if the following conditions are met.

(1) $\mathcal{A}$ satisfies the condition AB5 and is locally Artinian. That is, every object is generated by finite-length objects by filtered colimits. In particular, every object admit a (possibly infinite) composition series.

(2) There is an interval finite poset Λ (that is, the intervals $[\mu,\nu]=\{\lambda|\mu\leqslant\lambda\leqslant\nu\}\subset\Lambda$ are finite) with the set of all non-isomorphic simple objects $\mathrm{IC}_\lambda\in\mathcal{A}$ indexed by Λ.

(3) There is a collection of costandard objects ∇_λ indexed by Λ.

(4) These objects contain $\mathrm{IC}_\lambda\subset\nabla_\lambda$ as a subobject, and the quotient $\nabla_\lambda/\mathrm{IC}_\lambda$ contains only IC_μ with $\mu>\lambda$ in the composition series. Moreover, $[\nabla_\mu:\mathrm{IC}_\nu]$ and $\dim_k\mathrm{Hom}(\nabla_\mu,\nabla_\nu)$ are finite.

(5) IC_λ admit an injective envelope I_λ in $\mathcal{A}$. This injective I_λ admit another complete filtration

$$0=F_0\subset F_1\subset\cdots\subset I_\lambda$$

such that $F_1=\nabla_\lambda$, and $F_i/F_{i-1}=\nabla_\mu$ where $\mu>\lambda$ for $i>1$. Moreover, each ∇_μ appears only finitely many times.

This is not the main focus of this section, so we only present the main theorem of the article.

Theorem 5.1.1.2 (Theorem 3.9, [CPS88]). Finite length highest weight categories with finitely many simple objects are always of the form as a right module category $\mathbf{Mod}_A^{f.g.}$ for some right quasi-hereditary algebra A.

This is mainly used in the paper [BGS96]. In their paper, they only focused on finite length categories with finitely many simple objects, namely, the category of perverse sheaves with respect to a nice enough finite stratification. Their definitions went as follows.

Definition 5.1.1.3. Let $\mathcal{A}$ be an Abelian category over some field k. We say $\mathcal{A}$ is highest weight in the sense of [BGS96] if the following conditions hold.

(1) $\mathcal{A}$ is of finite length.

(2) $\mathcal{A}$ has simple objects indexed by a finite poset Λ, denoted by IC_λ. Moreover, $\mathrm{End}(\mathrm{IC}_\lambda)=k$. The poset Λ will be referred to as the weight poset.

(3) There are standard and costandard objects Δ_λ and ∇_λ indexed by Λ, equipped with structure morphisms $\Delta_\lambda\to\mathrm{IC}_\lambda$ and $\mathrm{IC}_\lambda\to\nabla_\lambda$.

(4) Let $\Lambda'\subset\Lambda$ be any subposet closed under taking smaller elements. We define $\mathcal{A}_{\Lambda'}$ to be the subcategory of $\mathcal{A}$ spanned by objects with simple components $\mathrm{IC}_{\lambda'}$ for $\lambda'\in\Lambda'$. Then, for any maximal element $m\in\Lambda'$, we have $\Delta_m\to\mathrm{IC}_m$ a projective cover and $\mathrm{IC}_m\to\nabla_m$ an injective hull in $\mathcal{A}_{\Lambda'}$.

(5) $\ker(\Delta_\lambda \to \mathrm{IC}_\lambda)$ and $\mathrm{coker}(\mathrm{IC}_\lambda \to \nabla_\lambda)$ lie in $\mathcal{A}_{<\lambda}$.
(6) $\mathrm{Ext}^2(\Delta_\mu, \nabla_\nu) = 0$.

Lemma 5.1.1.4. We have the following orthogonality and Ext vanishing conditions:

$$\dim_k \mathrm{Hom}(\Delta_\mu, \nabla_\nu) = \delta_{\mu,\nu}, \mathrm{Ext}^{i(\Delta_\mu,\nabla_\nu)} = 0$$

Proof: For the first orthogonality condition, note $\Delta_\mu \to \mathrm{IC}_\mu$ is a projective cover in $\mathcal{A}_{\leqslant\mu}$, so every quotient of Δ_μ must have IC_μ as a factor. Similarly, every subobject of ∇_ν must intersect IC_ν. Thus, if there is a non-trivial map $\Delta_\mu \to \nabla_\nu$, since the image and coimage is non-zero, we have

$$[\Delta_\mu : \mathrm{IC}_\nu] \neq 0, [\Delta_\nu : \mathrm{IC}_\mu] \neq 0$$

which implies $\nu \leqslant \mu$ and $\mu \leqslant \nu$, so $\mu = \nu$. Then, to compute $\dim_k \mathrm{Hom}(\Delta_\mu, \nabla_\mu)$, by the fifth axiom, we have

$$\mathrm{Hom}(\Delta_\mu, \nabla_\mu) = \mathrm{End}(\mathrm{IC}_\mu) = k$$

For the Ext^1 vanishing, we put $\Lambda' = \{\leqslant \mu\} \cup \{\leqslant \nu\}$. Then, either μ or ν or both is maximal in Λ', so the exact sequence

$$0 \to \nabla_\nu \to M \to \Delta_\mu \to 0$$

splits by either projectivity of ∇_ν or injectivity of Δ_μ in $\mathcal{A}_{\Lambda'}$.

For higher Ext vanishing, we prove by descending induction on μ. If μ is maximal in Λ, then Δ_μ is projective and there is nothing to prove. In general, consider $P_\mu \to \mathrm{IC}_\mu$, which has a standard filtration with Δ_μ the last term. Thus considering the long exact sequence of $\mathrm{Hom}(-, \nabla_\nu)$ applied to

$$0 \longrightarrow K \longrightarrow P_\mu \longrightarrow \Delta_\mu \longrightarrow 0$$

we have

$$\mathrm{Ext}^{i-1}(K, \nabla_\nu) \longrightarrow \mathrm{Ext}^i(\Delta_\mu, \nabla_\nu) \longrightarrow \mathrm{Ext}^i(P_\mu, \nabla_\nu)$$

exact with the first term and last term 0 by induction. Thus the claim follows. □

Proposition 5.1.1.5. Such categories have enough projective objects, with each projective admitting a finite filtration with standard associated graded pieces. The dual argument applies for injective objects.

Proof: We prove the following statement by induction: There are enough projectives in $\mathcal{A}_{\Lambda'}$, and each one admit a standard filtration.

We prove by induction on the cardinality of Λ'. The case where $\#\Lambda' = 1$ is trivial. Then, suppose Λ' and $\Lambda' \cup \{\lambda\}$ are both closed under taking smaller elements. We will prove the statement for $\Lambda' \cup \{\lambda\}$ from the statement for Λ'.

Now, for any $\mu \in \Lambda' \cup \{\lambda\}$, we attempt to construct a projective cover of IC_μ in $\mathcal{A}_{\Lambda' \cup \{\lambda\}}$. Since Λ' is closed under taking smaller elements, it is necessary that λ is an maximal element. Thus, $\Delta_\lambda \to \mathrm{IC}_\lambda$ is a desired projective cover. Then, for $\mu \in \Lambda'$, we wish to upgrade the projective cover $P \to \mathrm{IC}_\mu$ in $\mathcal{A}_{\Lambda'}$ to a projective cover in $\mathcal{A}_{\Lambda' \cup \{\lambda\}}$. We put $E = \mathrm{Ext}^1(P, \Delta_\lambda)$. Since P is not yet projective, E somehow measures the extend that P is non-projective. Then consider the unit element in

$$1 \in E \otimes E^* = \mathrm{Ext}^1(P, E^* \otimes \Delta_\lambda)$$

This gives rise to an extension

$$0 \longrightarrow E^* \otimes \Delta_\lambda \longrightarrow \tilde{P} \longrightarrow P \longrightarrow 0$$

such that the composition

$$E \to \mathrm{Hom}(E^* \otimes \Delta_\lambda, \Delta_\lambda) \xrightarrow{\delta} \mathrm{Ext}^1(P, \Delta_\lambda)$$

is the identity. Then, since the first map is an isomorphism because $\mathrm{End}(\Delta_\lambda) = \mathrm{End}(\mathrm{IC}_\lambda) = k$, the boundary map is also an isomorphism. Then, by projectivity of Δ_λ, $\mathrm{Ext}^1(\Delta_\lambda, \Delta_\lambda) = 0$. Hence, the long exact sequence

$$\mathrm{Hom}(E^* \otimes \Delta_\lambda, \Delta_\lambda) \longrightarrow \mathrm{Ext}^1(P, \Delta_\lambda) \longrightarrow \mathrm{Ext}^1\left(\tilde{P}, \Delta_\lambda\right) \longrightarrow \mathrm{Ext}^1(E^* \otimes \Delta_\lambda, \Delta_\lambda)$$

gives $\mathrm{Ext}^1\left(\tilde{P}, \Delta_\lambda\right) = 0$. Now, it suffices to prove that $\tilde{P}$ is a projective cover for IC_μ. This is equivalent to saying

$$\dim_k \mathrm{Hom}\left(\tilde{P}, \mathrm{IC}_\nu\right) = \delta_{\mu,\nu}, \mathrm{Ext}^1\left(\tilde{P}, \mathrm{IC}_\nu\right) = 0$$

where the second condition ensures that $\tilde{P}$ is projective and the first condition ensures that $\tilde{P} \to \mathrm{IC}_\mu$ is a cover. Consider the exact sequence

$$\mathrm{Ext}^1(P, \mathrm{IC}_\nu) \longrightarrow \mathrm{Ext}^1\left(\tilde{P}, \mathrm{IC}_\nu\right) \longrightarrow \mathrm{Ext}^1(E^* \otimes \Delta_\lambda, \mathrm{IC}_\nu)$$

the first term vanishes by the projectivity of P in $\mathcal{A}_{\Lambda'}$, and the third vanishes by projectivity of Δ_λ. So the second group vanishes. By the same reason, $\dim_k \mathrm{Hom}\left(\tilde{P}, \mathrm{IC}_\nu\right) = \delta_{\mu,\nu}$. Thus, it suffices to prove the statement for IC_λ. Then, to compute $\mathrm{Hom}\left(\tilde{P}, \mathrm{IC}_\lambda\right)$, consider the exact sequence

$$0 \longrightarrow K \longrightarrow \Delta_\lambda \longrightarrow \mathrm{IC}_\lambda \longrightarrow 0$$

Then, $\mathrm{Hom}\left(\tilde{P}, \mathrm{IC}_\lambda\right) = 0$ because K lies in $\mathcal{A}_{\Lambda'}$ and run the exact sequence, and Ext^1 vanishes by the vanishing condition of Ext^2.

Hence, the original statement follows from induction. □

Corollary 5.1.1.6. BGG reciprocity holds, that is, we have $\left(P_\mu : \Delta_\nu\right) = \left[\nabla_\nu : L_\mu\right]$

Proof: By the Hom orthogonality, the first number is equal to $\dim_k \mathrm{Hom}\left(P_\mu, \nabla_\nu\right)$. For the second, consider the Jordan-Holder series of ∇_ν

$$0 = F_0 \subset \cdots \subset F_n = \nabla_\nu$$

By the projectivity of P_μ, we have

$$0 \longrightarrow \mathrm{Hom}\left(P_\mu, F_{i-1}\right) \longrightarrow \mathrm{Hom}\left(P_\mu, F_i\right) \longrightarrow \mathrm{Hom}\left(P_\mu, \mathrm{IC}_\lambda\right) \longrightarrow 0$$

exact, where IC_λ is a simple factor. But then, since $P_\mu \to \mathrm{IC}_\mu$ is a projective cover, $P_\mu \to \mathrm{IC}_\lambda$ is nontrivial iff $\lambda = \mu$, so the second number is equal to $\dim_k \mathrm{Hom}(P_\mu, \nabla_\nu)$. □

Proposition 5.1.1.7. Let $X = \coprod_{\lambda\in\Lambda} X_\lambda$ be an affine stratification. Then, $\mathbf{Perv}_\Lambda(X)$ is a highest weight category in the sense of [BGS96].

Proof: We verify the axioms. The first three axioms are obtained directly by the properties of perverse sheaves on an affine stratified space. For the fourth property, it suffices to prove that $*$- or !-restriction to the open strata is exact (which is standard), because

$$\mathrm{Hom}(\Delta, -) = \mathrm{Hom}(j_! 1, -) = \mathrm{Hom}(1, j^*(-)) = j^*(-)$$

and the dual holds for ∇. Moreover, these objects are indecomposable, so they are the projective cover and injective hull. The fifth property also follows from the property of intermediate extensions. The final property is slightly more difficult, and is proven by the following property of perverse sheaves: firstly, by an analysis of extensions we have

$$\mathrm{Ext}^2_{\mathbf{Perv}_\Lambda}(\Delta_\mu, \nabla_\nu) \subset \mathrm{Hom}^2_{D_\Lambda}(\Delta_\mu, \nabla_\nu)$$

and for the latter, we have

$$\begin{aligned}\mathrm{Hom}^2_{D_\Lambda}(\Delta_\mu, \nabla_\nu) &= \mathrm{Hom}_{D_\Lambda}(j_{\mu !}1_\mu[\dim X_\mu], j_{\nu *}1_\nu[\dim X_\nu + 2]) \\ &= \mathrm{Hom}_{D_\Lambda}(j_\nu^* j_{\mu !}1_\mu[\dim X_\mu], 1_\nu[\dim X_\nu + 2])\end{aligned}$$

Then, the first term is 0 for $\nu \neq \mu$, and if $\nu = \mu$, then the above Hom is an ext group of vector spaces, which is 0. □

Corollary 5.1.1.8. We have the realization functor an equivalence.

$$D(\mathbf{Perv}_\Lambda(X)) \xrightarrow{\simeq} D_\Lambda(X)$$

Proof: Note that P_λ and ∇_λ separately generates both $D(\mathbf{Perv}_\Lambda(X))$ and $D_\Lambda(X)$. It suffices to show that

$$\mathrm{Ext}^i_{\mathbf{Perv}_\Lambda(X)}(P_\mu, \nabla_\nu) = \mathrm{Hom}^i_{D_\Lambda(X)}(P_\mu, \nabla_\nu)$$

By projectivity of P_μ, the left hand side vanishes unless $i = 0$. The right side also vanishes unless $i = 0$ by the standard filtration of P and applying the property of !-pushforwards. The statement for $i = 0$ is trivial. □

Now, we may wish to generalize the above properties to possibly infinite stratifications. This is where the modern definition of highest weight categories come in.

Definition 5.1.1.9. Let $\mathcal{A}$ be an Abelian category over some field k. We say $\mathcal{A}$ is highest weight in the sense of [Ric17] if it is highest weight in the sense of [BGS96], but with the condition of the finite poset Λ replaced by for any $\lambda \in \Lambda$, the set $\{\mu | \mu \leqslant \lambda\} \subset \Lambda$ is finite.

Properties such as the orthogonality and Ext vanishing still holds. However, the category of perverse sheaves over a stratified space may not be highest weight, for example

$$D(I\backslash LG/I) \neq D(\mathbf{Perv}(I\backslash LG/I))$$

which is an evidence of $\mathbf{Perv}(I\backslash LG/I)$ not being highest weight. Later on, we will adopt this definition of highest weight categories.

Lemma 5.1.1.10. Let $\Lambda' \subset \Lambda$ be a subposet closed under taking smaller objects, and let $\mathcal{A}$ be a highest weight category with weight poset Λ. Then, the Serre quotient $\pi : \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathcal{A}_{\Lambda'}$ is a highest weight category with weight poset $\Lambda - \Lambda'$. Its simple, standard, and costandard objects are the image of the projection π of the original simple, standard, and costandard objects.

Proof: Axioms (1), (2), (4), (5) are obvious. Then, for (4), we claim that we have an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\Delta_\lambda, N) = \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_{\Lambda'}}(\pi(\Delta_\lambda), \pi(N))$$

for $\lambda \in \Lambda - \Lambda'$. This is because a map $\pi(\Delta_\lambda) \to \pi(N)$ is represented by a map $M' \to N/N'$ in $\mathcal{A}$, where $M' \subset \Delta_\lambda$ and $N' \subset N$ such that Δ_λ/M and N' is in $\mathcal{A}_\Lambda$. Then, since every quotient of Δ_λ necessarily contains $\mathrm{IC}_\lambda \notin \mathcal{A}_\Lambda$, we have $M' = \Delta_\lambda$. Then, since $N' \in \mathcal{A}_{\Lambda'} \subset \mathcal{A}_{\leqslant\lambda}$, and that Δ_λ is projective in $\mathcal{A}_{\leqslant\lambda}$, we have $\mathrm{Ext}^1(\Delta_\lambda, N') = 0$, which implies that the map $\Delta_\lambda \to N/N'$ factors through $N \to N/N'$. Hence, we have the above map is surjective. Moreover since $M' = \Delta_\lambda$ necessarily, and any map $\Delta_\lambda \to N/N'$ does note annihilate IC_λ, it is non-zero. Thus the above map is an isomorphism. This implies that Δ_λ is projective in $(\mathcal{A}/\mathcal{A}_{\Lambda'})_{\Lambda''}$ for λ maixmal in Λ''.

Finally for axiom (6), it suffices to prove the case where Λ is finite becuase to compute $\mathrm{Ext}^2\big(\pi(\Delta_\mu), \pi(\nabla_\nu)\big)$, it suffices to compute this in the category $\mathcal{A}_{\leqslant\mu,\nu}$, which is finite by assumption. However, in the finite case, we have enough projectives, which allows us to prove the claim as follows. We consider the object $\pi(P_\mu) \twoheadrightarrow \pi(\Delta_\mu)$. The kernel of this map has a filtration by $\pi(\Delta_\lambda)$ where $\lambda > \mu$. Then, if $\pi(P_\mu)$ is projective, the above vanishing of Ext^2 follows by the long exact sequence given by

$$\mathrm{Ext}^1(K, \pi(\nabla_\nu)) \longrightarrow \mathrm{Ext}^2\big(\pi(\Delta_\mu), \pi(\nabla_\nu)\big) \longrightarrow \mathrm{Ext}^2\big(\pi(P_\mu), \pi(\nabla_\nu)\big)$$

where the last term is 0 by projectivity and the first is 0 by analysing extensions. Thus it suffices to prove projectivity. If not, consider a non-split surjection $\pi(M) \twoheadrightarrow \pi(P_\mu)$. This surjection is represented by a map $M' \to P_\mu/N'$ where M/M' and N' lies in $\mathcal{A}_{\Lambda'}$. Moreover, by surjectivity, this map must have cokernel lying in $\mathcal{A}_{\Lambda'}$. However, since $P_\mu \twoheadrightarrow \mathrm{IC}_\mu$ is a projective cover with $\mu \notin \Lambda'$, the cokernel must necessarily vanish, so the representing map is itself surjective. Thus, $M' \twoheadrightarrow P_\mu$ splits, which contradicts $\pi(M) \twoheadrightarrow \pi(P_\mu)$ not split. □

Proposition 5.1.1.11. Let $\mathcal{A}$ be a highest weight category with weight poset Λ. Then we have the following.

(1) The embedding $\iota : D(\mathcal{A}_{\Lambda'}) \hookrightarrow D(\mathcal{A})$ is fully faithful.
(2) The embedding $\iota : D(\mathcal{A}_{\Lambda'}) \hookrightarrow D(\mathcal{A})$ admit left and right adjoints, denoted by ι^L and ι^R. Moreover, the projection $\pi : D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{A}_{\Lambda'})$ admit left and right adjoints, denoted by π^L and π^R. These adjoints provide a semi-orthogonal decomposition of $D(\mathcal{A})$, namely

Definition 5.1.1.12. Tilting objects

todo 5.1.1.1. Finish this section.

5.1.2. Where is Koszul duality?

5.2. Koszul Duality for Kac–Moody groups

In this chapter, we give a review of Bezrukavnikov-Yun's paper [YB14]

5.2.1. Kac–Moody lie algebras and Kac–Moody groups

Definition 5.2.1.1. We define a generalized Cartan matrix of size ℓ to be a matrix $A \in \mathrm{Mat}_{\ell\times\ell}(\mathbb{Z})$, such that the following conditions are satisfied:
(1) $a_{i,i} = 2$;
(2) $a_{i,j} \leqslant 0$ for $i \neq j$;
(3) $a_{i,j} = 0$ iff $a_{j,i} = 0$.

We define a realization of a generalized Cartan matrix to be a triple $(\mathfrak{h}, \pi, \pi^\vee)$, where $\mathfrak{h}$ is a complex $(2\ell - \mathrm{rank}\, A)$-dimensional vector space, $\pi = \{\alpha_1, \cdots, \alpha_\ell\} \subset \mathfrak{h}^*$ a set of roots, and $\pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_\ell^\vee\} \subset \mathfrak{h}$ a set of coroots, subject to the relation $\alpha_j(\alpha_i^\vee) = a_{i,j}$.

By linear algebra, such realization exists and is unique up to isomorphism.

Construction 5.2.1.2. Given a generalized Cartan matrix A, we construct its associated Kac–Moody Lie algebra $\mathfrak{g}(A)$ by the following presentation:

- It is generated by $\mathfrak{h}$ and symbols $\{e_1, \cdots, e_\ell, f_1, \cdots, f_\ell\}$.
- The relations are given by
  (1) $[\mathfrak{h}, \mathfrak{h}] = 0$.
  (2) $[h, e_i] = \alpha_i(h)e_i$, $[h, f_i] = -\alpha_i(h)f_i$.
  (3) $\left[e_i, f_j\right] = \delta_{i,j}\alpha_i^\vee$.
  (4) $(\mathrm{ad}\, e_i)^{1-a_{i,j}}\left(e_j\right) = 0$ for $i \neq j$, and $(\mathrm{ad}\, f_i)^{1-a_{i,j}}\left(f_j\right) = 0$.

We denote $\mathfrak{n}$ and $\mathfrak{n}^-$ by the subalgebra generated by $\{e_i\}$ and $\{f_i\}$ respectively. We define $\mathfrak{b} = \mathfrak{n} + \mathfrak{h}$ and $\mathfrak{b}^- = \mathfrak{n}^- + \mathfrak{h}$ the standard Borel subalgebras. We also define $\tilde{\mathfrak{g}}$ to be the Lie algebra generated by the same generators but only subject to the relations (1), (2), and (3).

Definition 5.2.1.3. Let $\alpha \in \mathfrak{h}^*$. We define the root space $\mathfrak{g}_\alpha \subset \mathfrak{g}$ to be the subspace

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$$

We define $Q = \bigoplus \mathbb{Z}\alpha_i$, and $Q^+ = \bigoplus \mathbb{Z}_{\geqslant 0}\alpha_i$ the root lattice and nonnegative root lattice. Q^- is defined likewise. We say $\alpha \in Q^+$ is a positive root if $\mathfrak{g}_\alpha \neq 0$. We denote the set of positive roots by R^+. We likewise define R and R^-.

For a root α, we define $|\alpha| = \sum n_i$, where $\alpha = \sum n_i\alpha_i$.

Theorem 5.2.1.4. We have the following decompositions for $\mathfrak{g}$:

(1) $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ as vector spaces.

(2)
$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in Q^\pm} \mathfrak{g}_\alpha$$

Moreover, each root space is finite dimensional.

Proof: Omitted. □

Construction 5.2.1.5. Then, we slightly enlarge the above constructed Lie algebra. We define the complete version of the Lie algebra

$$\hat{\mathfrak{n}} = \prod_{\alpha \in R^+} \mathfrak{g}_\alpha$$

equipped with the natural Lie bracket. This is well-defined because each root space is finite dimensional. Then, notice the family of ideals

$$\hat{\mathfrak{n}}(k) = \prod_{\substack{\alpha \in R^+ \\ |\alpha| \geqslant k}} \mathfrak{g}_\alpha$$

gives $\hat{\mathfrak{n}}$ a structure of a pro-Lie algebra. Moreover, it is pro-nilpotent. Thus, its exponential gives a well-defined pro-algebraic group $\mathcal{U}$, namely given by

$$\mathcal{U} = \lim_k \exp(\hat{\mathfrak{n}}/\hat{\mathfrak{n}}(k))$$

Then, one defines the Kac–Moody group G by adding the weight 0 and negative part one by one, with no completion involved in the process.

However, the above construction has two caveats: firstly, it works well only for characteristic 0 fields. To work over base fields of arbitrary characteristic, we need to slightly modify the above and arrive at the following integral construction. Secondly, it lacks the additional root datum as in the finite dimensional case to define a group, and will arrive at only the simply connected Kac–Moody groups.

Definition 5.2.1.6. We define a Kac–Moody root datum of a generalized Cartan matrix A to be a finite rank free Abelian group X and morphisms of lattices $\alpha : Q \to X$ and $\alpha^\vee : Q^\vee \to X^\vee$, such that the pairing is preserved under the morphisms. Explicitly, we choose $\{\alpha_i\} \subset X$ and $\{\alpha_i^\vee\} \subset X^\vee$ such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$. We call this datum $\mathcal{D}$.

Construction 5.2.1.7 (Mathieu). We construct the Kac–Moody Lie algebra $\mathfrak{g}(\mathcal{D})$ associated to the Kac–Moody root datum $\mathcal{D}$. In fact, the construction is completely analogous to Construction 5.2.1.2, but with the naive $\mathfrak{h}$ replaced by $X^\vee \otimes \mathbb{Q}$, and everything works rationally. We will not copy the generators and relations again.

We then define the Chevalley–Tits integral enveloping algebra $\mathcal{U}_\mathbb{Z}(\mathfrak{g}) \subset U_\mathbb{Q}(\mathfrak{g})$. We take

$$e_i^{(n)} = \frac{e_i^n}{n!}, f_i^{(n)} = \frac{f_i^n}{n!}, \binom{x^\vee}{n} = \frac{x^\vee \cdot (x^\vee - 1) \cdot \cdots \cdot (x^\vee - n + 1)}{n!}$$

We then take $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g})$ to be the integral lattice spanned by the above generators.

We can likewise define $\mathcal{U}_{\mathbb{Z}}^{+} = \mathcal{U}_{\mathbb{Z}}(\mathfrak{n}^{+}) = U_{\mathbb{Q}}(\mathfrak{n}^{+}) \cap \mathcal{U}_{\mathbb{Z}}(\mathfrak{g})$, or equivalently generated by the elements $e_i^{(n)}$, by an integral version of the PBW theorem. More generally, take $\Psi \subset \Delta_{+}$ a subset of positive roots closed under addition. We take

$$\mathfrak{g}_{\Psi} = \bigoplus_{\alpha \in \Psi} \mathfrak{g}_{\alpha}$$

and $\mathcal{U}_{\mathbb{Z}}(\Psi) = U(\mathfrak{g}_{\Psi}) \cap \mathcal{U}_{\mathbb{Z}}(\mathfrak{g})$.

Then, we define the restricted gradewise dual of the above Hopf algebra

$$\mathbb{Z}[U_{\Psi}^{\max}] := \bigoplus_{\gamma \in \mathbb{N}\Psi} \mathcal{U}_{\mathbb{Z}}(\Psi)_{\gamma}^{\vee}$$

where $\mathcal{U}_{\mathbb{Z}}(\Psi)_{\gamma}$ denote the graded piece of $\mathcal{U}_{\mathbb{Z}}(\Psi)$ with weight γ. Then since $\mathcal{U}_{\mathbb{Z}}(\Psi)$ is a cocommutative Hopf algebra, we have $\mathbb{Z}[U_{\Psi}^{\max}]$ a commutative Hopf algebra, whose spectrum defines a group scheme

$$U_{\Psi}^{\max} = \operatorname{Spec} \mathbb{Z}[U_{\Psi}^{\max}]$$

We can see that this is a pro-unipotent group via the following height filtration:

$$U_{\Psi}^{\max} = \lim U_{\Psi}^{\max} / U_{\Psi, \geqslant n}^{\max}$$

where $(\Psi, \geqslant n)$ is the subset of Ψ containing of positive roots α such that $|\alpha| \geqslant n$. We can also compute its k-points for any k. Take

$$\widehat{\mathcal{U}}_k(\Psi) = \prod_{\gamma \in \mathbb{N}\Psi} (\mathcal{U}_{\mathbb{Z}}(\Psi) \otimes k)$$

and we have $U_{\Psi}^{\max}(k)$ the group-like elements in $\widehat{\mathcal{U}}_k(\Psi)$, which is another evidence of pro-unipotence. For $\Psi = \Delta_{+}$, we define $U_{+}^{\max} = U_{\Psi}^{\max}$.

Construction 5.2.1.8 (Mathieu). Now we construct the whole group. For this, we first need the exponential map. Let $x \in \mathfrak{g}_{\alpha,\mathbb{Z}}$. We define an exponential sequence for x to be $\left(x^{[n]}\right)_{n \geqslant 0}$, where $x^{[n]} \in \mathcal{U}_{\mathbb{Z}, n\alpha}^{+}$, subject to the conditions $x^{[0]} = 1$, $x^{[1]} = x$, $x^{[n]} - x^{(n)} \in F_{n-1} U_{\mathbb{Q}}(\mathfrak{g})$, where F is the PBW filtration (this means their difference is measured by polynomials of degree at most $n-1$).
Moreover, we require the formula

$$\Delta\left(x^{[n]}\right) = \sum_{a+b=n} x^{[a]} \otimes x^{[b]}, \varepsilon\left(x^{[n]}\right) = 0, n > 0$$

For any x, an exponential sequence always exists: firstly one finds integral lifts of $x^{(n)}$, and modulo low PBW filtered terms to adjust to the final formula.

Notice that for α a real root, we may choose $x^{[n]} = x^{(n)}$ because α is conjugate to some simple root α, so $x^{(n)}$ is known to be integral. The above formulas are obviously satisfied.

We now use this exponential sequence to construct an exponential map. Let R be any commutative ring. For $\lambda \in R$, we define $[\exp](\lambda x) := \sum \lambda^n x^{[n]} \in \widehat{\mathcal{U}}_R^{+}$. The above formulas ensure that $[\exp](\lambda x)$ is a group-like element, so lies in $U_{+}^{\max}(R)$. Then, this map is functorial in R, so assembles to a map $E_x : \mathbb{A}_{\mathbb{Z}}^1 \to U_{+}^{\max}$. Note that by unwinding definitions, this map is a group homomorphism iff

$$x^{[m]}x^{[n]} = \binom{m+n}{m} x^{[m+n]}$$

which holds for α a real root, but not necessarily holds for imaginary roots.

We take H to be the torus with root lattice X and coroot lattice $X^\vee$. Then, we have $H \circlearrowright U_+^{\max}$ via $h \cdot [\exp](\lambda x) \cdot h^{-1} = [\exp](\alpha(h)\lambda x)$ for $x \in \mathfrak{g}_\alpha$. We left it to the readers that this action is indeed well-defined. With this property, we may define

$$B_+^{\max} = U_+^{\max} \rtimes H$$

We then extend this definition to negative roots. For each simple root α_i, the $\mathfrak{sl}_2$-triple $e_i, f_i, \alpha_i^\vee$ integrates to a reductive group G_i with positive part U_i, and take $G_i *_{U_i} B_+^{\max}$ gives a minimal parabolic group $P_i \supset B_+^{\max}$, with the quotient $P_i/B_+^{\max}$ given by $\mathbb{P}^1$.

Then let W denote the Weyl group of the Kac–Moody algebra (i.e., the subgroup of $\mathrm{GL}(X \otimes \mathbb{Q})$ generated by the reflections $s_i : \alpha \mapsto \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i$). Take a word presentation $\boldsymbol{s} = s_{i_1} \cdot \cdots \cdot s_{i_n}$. We define $E(\boldsymbol{s}) = P_{i_1} \times^B P_{i_2} \times^B \cdots \times^B P_{i_n}$ (in this construction, we abbreviate $B_+^{\max}$ by B). Then, we define the Bott-Samuelson variety $D(\boldsymbol{s}) = E(\boldsymbol{s})/B$, where B acts on the right of P_{i_n}. If we take $\boldsymbol{s}' = s_{i_1} \cdot \cdots \cdot s_{i_{n-1}}$, we have that $D(\boldsymbol{s}) \twoheadrightarrow D(\boldsymbol{s}')$ by forgetting the final factor. This is actually a $\mathbb{P}^1$-bundle since $P_{i_n}/B = \mathbb{P}^1$. Then, $D(\varnothing) = \mathrm{pt}$, so $D(\boldsymbol{s})$ is smooth, projective, and geometrically integral. We remark here that $D(\boldsymbol{s})$ is the compactification of

$$Bs_{i_1}B \times^B Bs_{i_2}B \times^B \cdots \times^B B_{s_{i_n}}B/B = \mathbb{A}^n := D(w)^\circ$$

We then construct the Schubert cells. Take a regular dominant weight λ and its highest weight module $L(\lambda)$ (finite dimesional) with highest weight vector v_λ. Then, for $w \in W$ we pick a reduced word presentation (which we abuse notations and also denote by w). We have a map

$$\pi_w : D(w) \to \mathbb{P}(L(\lambda)), [p_1, \cdots, p_n] \mapsto [p_1 \cdot p_2 \cdot \cdots \cdot p_n \cdot v_\lambda]$$

The image is well-defined since B-action preserves the highest weight line in $L(\lambda)$. Then, picking a set of Weyl lifts $\dot{s}_{i_j}$, $\pi_w\left(\left[\dot{s}_{i_1}, \cdots, \dot{s}_{i_n}\right]\right)$ is the line $[L(\lambda)_{w\lambda}]$. We define $S_w = \overline{B[L(\lambda)_{w\lambda}]} \subset \mathbb{P}(E_w(\lambda))$ where $E_w(\lambda) = U(\mathfrak{b})L(\lambda)_{w\lambda}$ is upward part of $L(\lambda)$ over $L(\lambda)_{w\lambda}$ (sorry for the unfortunate notation). This image has the image of a projective scheme, induced by $\pi_{w,*}\mathcal{O}_{D(w)}$, hence we obtain a map of projective schemes $\pi_w : D(w) \twoheadrightarrow S_w$. After we have finished constructed G and G/B, we should think $S_w \approx \overline{BwB/B}$, and the projection $\pi_w : D(w) \twoheadrightarrow S_w$ as the multiplication map

$$P_{i_1} \times^B \cdots \times^B P_{i_n}/B \to \overline{BwB/B}$$

Moreover, we have π_w an isomorphism on the open cells $\pi_w : D(w)^\circ \xrightarrow{\simeq} C_w = \mathrm{im}(D(w)^\circ)$.

Finally, we construct $B(w)$, which is a B-torsor over S_w, which we should think of the closed cell $\overline{BwB}$. We first let $\mathcal{L}(M)$ be the constant quasi-coherent sheaf over $D(w)$ associated to the B-module M, and $\mathcal{L}_w(M) = \pi_{w,*}\mathcal{L}(M)$. We define $B(w) = \underline{\mathrm{Spec}}_{S_w} \mathcal{L}_w(k[B])$, which gives an affine morphism $\nu_w : B(w) \to S_w$. Actually, we have the non-trivial fact that $B(w) = \mathrm{Spec}\,\Gamma\left(E(w), \mathcal{O}_{E(w)}\right)$. However the core property we want to use is that $B(w) \to S_w$ is a B-torsor.

For $u, v \in W$, recall that we can define $u * v \in W$ to be the element defined recursively: pick a reduced word presentation $s_{i_1} \cdot \cdots \cdot s_{i_r} = v$, we define $u * v := u * s_{i_1} * s_{i_2} * \cdots * s_{i_r}$, and for a simple reflection s, we define $u * s = \max(u, us)$. Then, the above construction gives a multiplication $E(u) \times E(v) \to B(u * v)$, which descends to $B(u) \times B(v) \to B(u * v)$ (to be thought of $\overline{BuB} \times \overline{BvB} \subset \overline{B(u * v)B}$) by affinization. With these structural maps, we take

$$G = \operatorname*{colim}_{w \in W} B(w)$$

which gives a ind-group structure.

Construction 5.2.1.9. We define $\mathrm{Fl} = G/B = \operatorname{colim} S_w$ to be the affine flag variety, which is also canonically an ind-scheme. The open Schubert cell $\mathrm{Fl}_w = BwB/B = \mathbb{A}^{\ell(w)}$, which can also be thought of the image of C_w constructed previously. We hence have

$$\mathrm{Fl} = \coprod_{w \in W} \mathrm{Fl}_w, S_w = \coprod_{v \leqslant w} \mathrm{Fl}_w$$

Likewise, we may define the enhanced flag variety $\widetilde{\mathrm{Fl}} = G/U$, which is a H-torsor over $\mathrm{Fl} = G/B$.

Construction 5.2.1.10. If we let $\Sigma \subset W$ denote the subset of simple reflections, and take $\Theta \subset \Sigma$ such that the subgroup of W generated by Θ is finite, likewise to the construction of P_i's in Construction 5.2.1.8 we can construct a parabolic subgroup P_Θ. Then, we define L_Θ the Levi subgroup, and U_Θ the unipotent radical of P_Θ. We let B_Θ be the Borel of L_Θ. Finally, we define $U^-_{L_\Theta}$ to be the negative unipotent radical of L_Θ.

In the following chapter, we will adopt the left flag variety notation. That is, we take $\mathrm{Fl} = B\backslash G$, and for any Θ satisfying the condition above, take ${}_\Theta\mathrm{Fl} = P_\Theta\backslash G$. Moreover, we let W_Θ denote the Weyl group of L_Θ, and $[W_\Theta\backslash W] \subset W$ the set of minimal length representatives of the cosets. It inherits a length function and a poset structure from W. We denote the maximal length element $w_\Theta \in W_\Theta$ of length ℓ_Θ. Then, we have the strata of $P_\Theta\backslash G/B$ indexed by $\overline{w} \in [W_\Theta\backslash W]$. We make the following definitions

$$i_{\overline{w}} : {}_\Theta\mathrm{Fl}_{\overline{w}} \hookrightarrow {}_\Theta\mathrm{Fl}, \qquad i_{<\overline{w}} : {}_\Theta\mathrm{Fl}_{<\overline{w}} \hookrightarrow {}_\Theta\mathrm{Fl}, \qquad i_{\leqslant\overline{w}} : {}_\Theta\mathrm{Fl}_{\leqslant\overline{w}} \hookrightarrow {}_\Theta\mathrm{Fl}$$

The stratas are affine spaces of corresopnding dimensions. When $\Theta = \varnothing$, this reduces to the plain flag variety case.

Then, consider $U_\Theta U^-_{L_\Theta}$. The orbits of $U_\Theta U^-_{L_\Theta}\backslash G/B$ are still indexed by $w \in W$, and we take

$${}_\Theta\mathrm{Fl}^w = U_\Theta U^-_{L_\Theta} \cdot BwB/B \subset {}_\Theta\mathrm{Fl}, \qquad {}_\Theta\mathrm{Fl}^{\overset{\Theta}{\leqslant} w} = \overline{{}_\Theta\mathrm{Fl}^w}$$

where $s \overset{\Theta}{\leqslant} t$ means $w_\Theta s \leqslant w_\Theta t$.

Finally, we take $\widetilde{\mathrm{Fl}} = G/U$, and make the likewise definitions $\tilde{\imath}_w : \widetilde{\mathrm{Fl}}_w \hookrightarrow \widetilde{\mathrm{Fl}}$ to be the preimages of Fl_w. The other definitions are done likewise.

Construction 5.2.1.11. We construct the big cell $C \subset \mathrm{Fl}$. In the finite dimensional case, this construction coincides with the cell B^-B/B. Again we construct it as an ind-scheme. Recall that we have a projective embedding $S_w \hookrightarrow \mathbb{P}(E_w(\lambda))$, where $E_w(\lambda) = U(\mathfrak{b})L(\lambda)_{w\lambda}$. Then, the contragradient $L(\lambda)^*$ has a lowest weight line $\sigma_{-\lambda}$. We define C_w to be the complement to the hyperplane $\sigma_{-\lambda} = 0$, and $C = \operatorname{colim} C_w$.

Lemma 5.2.1.12. Let $\mu : \mathbb{G}_m \to H$ be any antidominant regular coweight. Then for any $w \in W$, C_w contracts to the base point $B/B \in \mathrm{Fl}$ under the left action of $\mu(\mathbb{G}_m)$. Hence, for any $v, w \in W$, $vC \cap \mathrm{Fl}_{\leqslant w}$ contracts to the point $vB/B \in \mathrm{Fl}$ under the action of $(v\mu)\mathbb{G}_m$.

Proof: By definition of C_w. $\square$

5.2.2. A realization result

Definition 5.2.2.1. Consider the category $\mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$. We define a Frobenius module M to be an object $M \in \mathbf{Mod}_{\overline{\mathbb{Q}}_\ell}$ equipped with an automorphism $\mathrm{Fr}_M : M \to M$. We denote the category of Frobenius modules by $\mathbf{Mod}(\mathrm{Fr})$. We define a Frobenius module to be locally finite if it is a union of its finite submodules. For any Frobenius module M, we denote M^f by the union of its finite submodules.

There is clearly a forgetful functor $\mathbf{Mod}(\mathrm{Fr}) \to \mathbf{Vect}_{\overline{\mathbb{Q}}_\ell}$ given by $M \mapsto M^{\mathrm{Fr}}$, the fixed point functor. Later on, we will use this functor to realize a category enriched over $\mathbf{Mod}(\mathrm{Fr})$ to a DG category over $\overline{\mathbb{Q}}_\ell$.

Definition 5.2.2.2. We define a locally finite Frobenius module to be continuous if the action $\mathbb{Z} \circlearrowleft M$ through Frobenius extends to an action $\widehat{\mathbb{Z}} \circlearrowleft M$. For a locally finite Frobenius module, we may define its weights as in the usual convention. For a general Frobenius module M, we make the above definitions after applying M^f.

From now on, we assume every Frobenius module to be continuous with integral weights.

Definition 5.2.2.3. Let M be a Frobenius module. We define $M^{\mathrm{Fr}\text{-unip}}$ to be the Frobenius submodule of M on which Fr_M acts unipotently. We define M^{op} to denote the Frobenius module with the same underlying vector space as M, but with the inverse Frobenius action.

Definition 5.2.2.4. Let E be a $\overline{\mathbb{Q}}_\ell$-algebra. Let E be equipped with a continuous Frobenius module structure compatible with its algebra structure, we define $\mathbf{Mod}_{E,\mathrm{Fr}}$ to be the category of E-modules equipped with a Frobenius automorphism compatible with the Frobenius module structure on E.

In this section, our main goal is to answer the following question.

Question 5.2.2.5. For a category $\mathcal{D}$ enriched in $\mathbf{Mod}_{\mathrm{Fr}}$, when do we have a fully faithful inclusion $\mathcal{D} \hookrightarrow \mathbf{Mod}_{E,\mathrm{Fr}}$?

We will show this with the following assumptions.

Assumption 5.2.2.6. Let $\mathcal{D}$ to be a $\overline{\mathbb{Q}}_\ell$-linear category enriched in $D(\mathrm{Fr})$ (equivalently, the Hom and Ext groups are Frobenius modules), and is equipped with the following data:

(1) A finite poset Λ such that $\mathcal{D}$ has a recollement indexed by Λ (that is, a series of open-closed semi-orthogonal decompositions). We define $\mathcal{D}_\lambda = \mathcal{D}_{\leqslant\lambda}/\mathcal{D}_{<\lambda}$.

(2) A full additive subcategory $\mathcal{C}_\lambda \subset \mathcal{D}_\lambda$ stable under taking tensor with unipotent Frobenius modules, such that for any objects $c_1, c_2 \in \mathcal{C}_\lambda$, we have

$$\operatorname{Ext}^i_{\mathcal{D}_\lambda}(c_1, c_2)^{\text{Fr-unip}} = 0$$

(3) $\mathcal{C}_\lambda$ generates $\mathcal{D}_\lambda$ as a triangulated category. Moreover, there is an object $\mathbb{1}_\lambda \in \mathcal{C}_\lambda$ such that every object in $\mathcal{C}_\lambda$ is of the form $\mathbb{1}_\lambda \otimes M$ for some $M \in D(\mathrm{Fr})$ (however, for arbitrary M, the object $\mathbb{1}_\lambda \otimes M \in \mathcal{D}_\lambda$ may not lie in $\mathcal{C}_\lambda$).

(4) There exists an object $c_\lambda \in \mathcal{C}_{\leqslant\lambda}$ such that $i^*_\lambda c_\lambda = \mathbb{1}_\lambda$. Moreover, the induced ring morphism

$$i^*_\alpha : \operatorname{End}(c_\lambda) \to \operatorname{End}(\mathbb{1}_\lambda)$$

has a nilpotent kernel.

We let $\mathcal{C}$ denote the full subcategory of $\mathcal{D}$ consisting of objects d such that $i^*_\lambda d, i^!_\lambda d \in \mathcal{C}_\lambda$.

In practice, $\mathcal{D}$ will be a category of mixed ℓ-adic sheaves constructible with respect to a certain stratification, and $\mathcal{C}_\alpha$ the weight 0 constant sheaves (normally we would have locally constant, but we require constant in our case) over a strata. Then $\mathcal{C}$ will be the category of weight 0 sheaves.

The main theorem is the following, which would be our main technical tool later on.

Theorem 5.2.2.7. For a category $\mathcal{D}$ satisfying the requirements in Assumption 5.2.2.6, there is a $\overline{\mathbb{Q}}_\ell$-algebra E with a Frobenius action, such that $\mathcal{D} \hookrightarrow D(E, \mathrm{Fr})$ is fully faithful in the sense that both categories are enriched in **Set** rather than $D(\mathrm{Fr})$. Moreover, such construction is functorial for functors preserving the subcategory $\mathcal{C}$.

We will begin with a simple lemma on the computation of a homotopy fixed point.

Lemma 5.2.2.8. Let $M \in D(\mathrm{Fr})$. Then,

$$0 \longrightarrow H^{i-1}(M)_{\mathrm{Fr}} \longrightarrow H^i\left(M^{h\,\mathrm{Fr}}\right) \longrightarrow H^i(M)^{\mathrm{Fr}} \longrightarrow 0$$

Proof: Consider the following description of homotopy fixed points of a $\mathbb{Z}$ action as a fiber

$$M^{h\,\mathrm{Fr}} \longrightarrow M \xrightarrow{\mathrm{Fr}-\mathrm{id}} M$$

Then the desired exact sequence follows from the long exact sequence of the fiber sequence. □

Corollary 5.2.2.9. Let $\mathcal{D}$ satisfy the condition (1) and (2) in Assumption 5.2.2.6. Then, for $c_1, c_2 \in \mathcal{C}$, we have

$$\operatorname{Ext}^i_{\mathcal{D}}(c_1, c_2)^{\text{Fr-unip}} = 0, i \neq 0$$

and

$$H^i\Big(R\mathrm{Hom}_{\mathcal{D}}(c_1,c_2)^{h\,\mathrm{Fr}}\Big) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(c_1,c_2)^{\mathrm{Fr}} & \text{if } i=0 \\ \mathrm{Hom}_{\mathcal{C}}(c_1,c_2)_{\mathrm{Fr}} & \text{if } i=1 \\ 0 & \text{otherwise} \end{cases}$$

Proof: For the first assertion, we apply induction on the stratas. On one strata, this is precisely condition (2). Then consider the excision triangle, which gives rise to the following long exact sequence for $c_1, c_2 \in \mathcal{C}_{\leqslant\lambda}$

$$\cdots \longrightarrow \mathrm{Ext}^i_{\mathcal{C}_{<\lambda}}\big(i^*_{<\lambda}c_1, i^!_{<\lambda}c_2\big)^{\mathrm{Fr}\text{-unip}} \longrightarrow \mathrm{Ext}^i(c_1,c_2)^{\mathrm{Fr}\text{-unip}} \longrightarrow \mathrm{Ext}^i_{\mathcal{C}_\lambda}(i^*_\lambda c_1, i^*_\lambda c_2)^{\mathrm{Fr}\text{-unip}} \longrightarrow \cdots$$

which gives the vanishing result.

For the second assertion, notice that for any $M \in \mathbf{Mod}_{\mathrm{Fr}}$, we have $M^{\mathrm{Fr}} \subset M^{\mathrm{Fr}\text{-unip}}$. The rest follows from the previous lemma. □

Lemma 5.2.2.10. We adopt the decreasing filtration convention. Then, there is an equivalence of categories

$$\widetilde{\mathrm{gr}} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Ch}(\mathcal{C})$$

where $\mathrm{Fil}(\mathcal{C})$ is the full subcategory of $\mathrm{Fil}(\mathcal{D})$ such that $\mathrm{gr}^i(F^\bullet d) \in \mathcal{C}[-i]$, and $\mathrm{Ch}(\mathcal{C})$ is the category of chain complexes without any homotopy relations. Here, chain complexes mean the composite of two differentials are homotopic to 0. The equivalence is given by

$$\widetilde{\mathrm{gr}}(F^\bullet d) = \cdots \longrightarrow \mathrm{gr}^i(d)[i] \longrightarrow \mathrm{gr}^{i+1}(d)[i+1] \longrightarrow \cdots$$

where the transition maps are given by the following triangle

$$\mathrm{gr}^{i+1}(F^\bullet d) \longrightarrow \mathrm{gr}^{[i,i+1]}(F^\bullet d) \longrightarrow \mathrm{gr}^i(F^\bullet d) \longrightarrow \mathrm{gr}^{i+1}(F^\bullet d)[1]$$

Proof: This is well-known if $\mathcal{C}$ is a stable ∞-category. Now, $\mathcal{C}$ is a additive subcategory of such, the proof is likewise. □

Definition 5.2.2.11. Let $K_{\mathrm{acyc}}(\mathcal{C}) \subset K(\mathcal{C})$ denote the subcategory of the bounded homotopy category spanned by objects equivalent to $0 \in \mathcal{D}$.

Lemma 5.2.2.12. For any $c \in \mathcal{C}$, the functors

$$\mathrm{Hom}(c,-)^{\mathrm{Fr}\text{-unip}}, \mathrm{Hom}(-,c)^{\mathrm{Fr}\text{-unip}} : K_{\mathrm{acyc}}(\mathcal{C}) \to K_{\mathrm{acyc}}(\mathcal{C})$$

maps any complex to an exact sequence.

Proof: Fix an object $\mathcal{F} \in \mathrm{Fil}(\mathcal{C})$ such that $\widetilde{\mathrm{gr}}\,\mathcal{F} \in K_{\mathrm{acyc}}(\mathcal{C})$. Consider the spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^{p+q}(c, \mathrm{gr}^p\,\mathcal{F}) = \mathrm{Ext}^q(c, \mathrm{gr}^p\,\mathcal{F}[p]) \Longrightarrow \mathrm{Ext}^{p+q}(c, \mathcal{F})$$

However, since $\widetilde{\mathrm{gr}}\,\mathcal{F} \in K_{\mathrm{acyc}}(\mathcal{C})$, $\mathcal{F} \simeq 0 \in \mathcal{D}$. Thus, the above spectral sequence converges to 0. Then, applying $(-)^{\mathrm{Fr\text{-}unip}}$ to it, the spectral sequence degenerates at page 1 because q is forced to be 0. Thus, since the spectral sequence also converges to zero, the complex at $q = 0$ is exact. $\square$

Definition 5.2.2.13. We define $\mathcal{C}^{\mathrm{Fr\text{-}unip}}$ to be the category with the same objects as $\mathcal{C}$, but with Hom groups replaced by

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{Fr\text{-}unip}}}(c_1, c_2) = \mathrm{Hom}_{\mathcal{C}}(c_1, c_2)^{\mathrm{Fr\text{-}unip}}$$

Construction 5.2.2.14. Notice that the forgetful functor $\mathcal{C} \to \mathcal{C}^{\mathrm{Fr\text{-}unip}}$ admits a pro-left adjoint $\mathcal{C}^{\mathrm{Fr\text{-}unip}} \to \mathrm{Pro}(\mathcal{C})$, given by $c \mapsto c[[t]] = \lim c[t]/t^n = c \otimes \overline{\mathbb{Q}}_\ell[[t]]$, where Frobenius acts on $\overline{\mathbb{Q}}_\ell[[t]]$ by multiplying by $\exp t$. Explicitly, we have

$$\mathrm{Hom}(c, c')^{\mathrm{Fr\text{-}unip}} = \mathrm{Hom}(c[[t]], c')^{\mathrm{Fr}}$$

We remark that the exponential here is used to ensure unipotency.

Proposition 5.2.2.15. We have a well-defined functor $\rho(\mathcal{C}) : K(\mathcal{C})/K_{\mathrm{acyc}}(\mathcal{C}) \to \mathcal{D}$. Moreover, it is fully faithful, and is essentially surjective if $\mathcal{C}_\alpha$ generates $\mathcal{D}_\alpha$ as a triangulated category.

Proof: Recall that maps $\mathrm{Hom}_{K(\mathcal{C})/K_{\mathrm{acyc}}(\mathcal{C})}(c_1, c_2)$ are computed by spans $c_1 \leftarrow c \to c_2$, where $\mathrm{cofib}(c \to c_1) \in K_{\mathrm{acyc}}(\mathcal{C})$. We may assume that $c_1 \in \mathcal{C}$, and we first show that taking the two term complex $c_1[[t]] \to c_1[[t]]$ is an initial (pro-) object mapping to c_1 with an acyclic cone. For this, we suppose $c \to c_1$ has an acyclic cone. Writing

$$c = \cdots \longrightarrow c^{-1} \longrightarrow c^0 \longrightarrow c^1 \longrightarrow \cdots$$

the cone is of the form

$$\cdots \longrightarrow c^{-1} \longrightarrow c^0 \longrightarrow c^1 \oplus c_1 \longrightarrow \cdots$$

Thus, by Lemma 5.2.2.12, taking $\mathrm{Hom}(c_1, -)^{\mathrm{Fr\text{-}unip}}$, we obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(c_1, c^0)^{\mathrm{Fr\text{-}unip}} \longrightarrow \mathrm{Hom}(c, c^1 \oplus c_1)^{\mathrm{Fr\text{-}unip}} \longrightarrow \cdots$$

This amounts to the exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(c_1[[t]], c^0)^{\mathrm{Fr}} \longrightarrow \mathrm{Hom}(c_1[[t]], c^1 \oplus c_1)^{\mathrm{Fr}} \longrightarrow \cdots$$

Note the Frobenius equivariant map $(0, p) : c_1[[t]] \to c^1 \oplus c_1$ has zero image, thus it comes from a map $\varphi_0 : c_1[[t]] \to c^0$. Then, since we want $- \times t : c_1[[t]] \to c_1[[t]]$, we also have $t\varphi_0 : c_1[[t]] \to c^0$ zero, so it comes from a map $\varphi^{-1} : c_1[[t]] \to c^{-1}$. Thus, we obtain a commutative square

$$\begin{tikzcd} c_1[[t]] \arrow[r, "t"] \arrow[d, "\varphi_{-1}"'] & c_1[[t]] \arrow[d, "\varphi_0"] \\ c^{-1} \arrow[r] & c^0 \end{tikzcd}$$

which in turn exhibits initialness.

Hence, $\mathrm{Hom}^\bullet_{K(\mathcal{C})/K_{\mathrm{acyc}}(\mathcal{C})}(c_1,c_2)^{h\,\mathrm{Fr}}$ is equivalently the two term complex

$$0 \longrightarrow \mathrm{Hom}(c_1[[t]],c_2)^{\mathrm{Fr}} \overset{(\times t)^*}{\longrightarrow} \mathrm{Hom}(c_1[[t]],c_2)^{\mathrm{Fr}} \longrightarrow 0$$

which, by the universal property, is the complex

$$0 \longrightarrow \mathrm{Hom}(c_1,c_2)^{\mathrm{Fr}\text{-unip}} \overset{\log \mathrm{Fr}}{\longrightarrow} \mathrm{Hom}(c_1,c_2)^{\mathrm{Fr}\text{-unip}} \longrightarrow 0$$

which is precisely the complex computing $\mathrm{Hom}^\bullet_{\mathcal{D}}(c_1,c_2)^{\mathrm{Fr}}$, by Corollary 5.2.2.9. Thus the functor is fully faithful. The image is clearly generated by the $\mathcal{C}_\alpha$'s, in a somewhat inaccurate sense. □

Now, we notice that $\mathcal{C}^{\mathrm{Fr}\text{-unip}}$ is in some sense the Frobenius semisimplification, as opposed to $\mathcal{C}^{\mathrm{Fr}}$ is removing the Frobenius structure. This can be shown in the following lemma.

Lemma 5.2.2.16. From now on, we require $\mathcal{D}$ to satisfy all assumptions from Assumption 5.2.2.6. For any $c \in \mathcal{C}$, there are complexes $M_\lambda \in D(\mathrm{Fr})$, such that

$$c = \bigoplus c_\lambda \otimes M_\lambda \in \mathcal{C}^{\mathrm{Fr}\text{-unip}}$$

Proof: Notice that we have $\mathrm{Ext}^1_{\mathcal{C}_{<\lambda}}\left(i^*_{<\lambda}c_1, i^!_{<\lambda}c_2\right)^{\mathrm{Fr}\text{-unip}} = 0$, so the restriction map

$$\mathrm{Hom}_{\mathcal{C}}(c_1,c_2)^{\mathrm{Fr}\text{-unip}} \twoheadrightarrow \mathrm{Hom}_{\mathcal{C}_\lambda}(i^*_\lambda c_1, i^*_\lambda c_2)^{\mathrm{Fr}\text{-unip}}$$

is surjective.

To prove the statement, we apply induction on the support of c. Suppose $c \in \mathcal{C}_{\leqslant\lambda}$. Then we have $i^*_\lambda c = \mathbb{1}_\lambda \otimes M_\lambda$. Moreover, since the left isomorphism is naturally Frobenius invariant, by the above surjection, the isomorphism lifts to a retraction diagram

$$c_\lambda \otimes M_\lambda \to c \to c_\lambda \otimes M_\lambda$$

This is a retraction since $\mathrm{End}(c_\lambda) \to \mathrm{End}(\mathbb{1}_\lambda)$ has nilpotent kernel, so any preimage of an isomorphism is an isomorphism.

Since retractions splits off, we may assume $c = c_\lambda \otimes M_\lambda \oplus c'$, where c' is supported on $\mathcal{C}_{<\lambda}$. Thus by induction, the proof is finished. □

Corollary 5.2.2.17. Let $c \in \mathcal{C}$ be such that $\mathrm{End}(c)$ is Fr-semisimple. Then,

$$c \simeq \bigoplus c_\lambda \otimes M_\lambda \in \mathcal{C}$$

such that M_λ is a Frobenius semisimple complex.

Proof: The idempotents corresponding to $c_\lambda \otimes M_\lambda$ in $\mathcal{C}^{\mathrm{Fr}\text{-unip}}$ is actually Frobenius invariant because $\mathrm{End}(c)$ is semisimple, so the direct sum decomposition occurs in $\mathcal{C}$. Then, obviously each M_λ would be semisimple, otherwise $\mathrm{End}(M_\lambda)$ would not be semisimple. □

Now, suppose there is a family of objects $d_\lambda \in \mathcal{C}_{\leqslant\lambda}$ also pulling back to $\mathbb{1}_\lambda$. For example, $d_\lambda = c_\lambda$ is a valid choice.

Lemma 5.2.2.18. $\mathcal{D}$ is generated by $\{d_\lambda\}_{\lambda\in\Lambda}$.

Proof: Again we apply induction on the stratas. Suppose $\mathcal{D}_{<\lambda}$ is generated by d_μs for $\mu < \lambda$. Then, by adjunction, we have a map $i_{\lambda!}\mathbb{1}_\lambda \to d_\lambda$, whose cofiber lies in $\mathcal{D}_{<\lambda}$, so is generated by d_μs for $\mu < \lambda$. Thus, $i_{\lambda!}\mathbb{1}$ is generated by d_μs for $\mu \leqslant \lambda$. But $\mathcal{D}_{\leqslant\lambda}$ is generated by $i_{\lambda!}\mathbb{1}_\lambda$ and $\mathcal{D}_{<\lambda}$ by assumption (3). Thus, we are done. □

Proof: Now we are ready to prove Theorem 5.2.2.7. We let $d = \bigoplus d_\lambda$, and take the $\overline{\mathbb{Q}}_\ell$-algebra E to be

$$E = \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_{\mathcal{D}}(d,d)^{\mathrm{op}}$$

In particular, this is concentrated at degree 0, so not a DG algebra. We will prove that we have a fully faithful functor $\mathcal{C} \hookrightarrow \mathbf{Mod}_{E,\mathrm{Fr}}$, and then take derived categories, and prove $K(\mathcal{C})/K_{\mathrm{acyc}}(\mathcal{C}) \hookrightarrow D(E,\mathrm{Fr})$ is fully faithful. Finally, by Proposition 5.2.2.15, we obtain the desired fully faithful map.

Firstly, the functor $\mathcal{C} \hookrightarrow \mathbf{Mod}_{E,\mathrm{Fr}}$ is via the obvious Yoneda functor

$$h_d = \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_{\mathcal{D}}(d,-) : \mathcal{C} \to \mathbf{Mod}_{E,\mathrm{Fr}}$$

Note that by definition

$$\mathrm{Hom}_{\mathbf{Mod}_{E,\mathrm{Fr}}}(h_d(c_1), h_d(c_2)) = \mathrm{Hom}_E(h_d(c_1), h_d(c_2))^{\mathrm{Fr}}$$

so we only need

$$\mathrm{Hom}_{\mathcal{C}}(c_1,c_2)^{\mathrm{Fr\text{-}unip}} \xrightarrow{\simeq} \mathrm{Hom}_E(h_d(c_1), h_d(c_2))^{\mathrm{Fr\text{-}unip}}$$

For this, if we have $c_1 = d$, then

$$\mathrm{Hom}_E(h_d(d), h_d(c_2))^{\mathrm{Fr\text{-}unip}} = \mathrm{Hom}_E(E, h_d(c_2))^{\mathrm{Fr\text{-}unip}} = h_d(c_2)^{\mathrm{Fr\text{-}unip}} = \mathrm{Hom}_{\mathcal{C}}(d,c_2)^{\mathrm{Fr\text{-}unip}}$$

And then, by projecting to direct summands in $\mathcal{C}^{\mathrm{Fr\text{-}unip}}$, we see the above map is an isomorphism for d_λs. Finally, since the d_λs generates $\mathcal{C}^{\mathrm{Fr\text{-}unip}}$ under direct sums, we have the functor fully faithful.

Then passing to homotopy categories, we first prove $K_{\mathrm{acyc}}(\mathcal{C})$ is null-homotopic in $K(\mathcal{C}^{\mathrm{Fr\text{-}unip}})$. This is done by Lemma 5.2.2.12, and explicitly constructing the contracting homotopy via the exactness of Hom functors. Thus, we have a well-defined map $K(\mathcal{C})/K_{\mathrm{acyc}}(\mathcal{C}) \to D(E,\mathrm{Fr})$. However, in this case, the Ext groups are computed likewise as above, so we are done.

Finally, for the functoriality, for a map $\Phi : \mathcal{D} \to \mathcal{D}'$ with $\Phi|_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}'$, we claim that we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{D} \arrow[d, "\Phi"'] & K(\mathcal{C})/K_{\mathrm{acyc}}(\mathcal{C}) \arrow[l, "\rho(\mathcal{C})"'] \arrow[r, "h_d"] \arrow[d, "K(\Phi|_{\mathcal{C}})"'] & D(E,\mathrm{Fr}) \arrow[d, "B_\Phi \otimes_E (-)"] \\
\mathcal{D}' & K(\mathcal{C}')/K_{\mathrm{acyc}}(\mathcal{C}') \arrow[l, "\rho(\mathcal{C}')"] \arrow[r, "h_{d'}"'] & D(E',\mathrm{Fr})
\end{tikzcd}$$

where $B_\Phi = \mathrm{Hom}_{\mathcal{C}'}(d', \Phi(d))$. The left square obviously commutes, and for the right square, it suffices to show the following square commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h_d"] \arrow[d, "\Phi|_{\mathcal{C}}"'] & \mathbf{Mod}_{E,\mathrm{Fr}} \arrow[d, "B_\Phi \otimes_E (-)"] \\ \mathcal{C}' \arrow[r, "h_{d'}"'] & \mathbf{Mod}_{E',\mathrm{Fr}} \end{tikzcd}$$

for this square, note we have a natural transformation

$$B_\Phi \otimes_E (-) = \mathrm{Hom}_{\mathcal{C}'}(d', \Phi(d)) \otimes_E \mathrm{Hom}_{\mathcal{C}}(d, -) \to \mathrm{Hom}_{\mathcal{C}'}(d', \Phi(-))$$

via the composition map. It suffices to prove this map is an isomorphism for all $c \in \mathcal{C}$. Obviously, it is an isomorphism for $c = d$ hence d_λ for any λ. By Lemma 5.2.2.16, c_λ is a direct summand of d_λ in $\mathcal{C}^{\mathrm{Fr\text{-}unip}}$, hence the above map is an isomorphism for all c_λ. But since c_λ generates the whole category as in Lemma 5.2.2.16, we are done. □

Finally, we will also be able to compute the image of the above fully faithful functor.

Corollary 5.2.2.19. The image of the functor $\mathcal{D} \hookrightarrow \mathbf{Mod}(E, \mathrm{Fr})$ is the full subcategory generated by the twists of $\mathrm{Hom}(d, d_\lambda)$s as a triangulated category.

Proof: Obvious. □

Another first corollary is the computation of A-equivariant derived category over a point, where A is any torus.

Definition 5.2.2.20. Let A be any torus over a finite field. Then we define its ℓ-adic Tate module

$$T_\ell(A) = \lim_{n \to \infty} A[\ell^n]$$

where $A[\ell^n]$ is the ℓ^n-torsion points of A, and $V_A = T_\ell(A) \times_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell = H_1\left(A, \overline{\mathbb{Q}}_\ell\right)$. By definition, this is a Fr-module of weight -2.

We then define $\check{S}_A = \mathrm{Sym}\, V_A^\vee$, which is a $\overline{\mathbb{Q}}_\ell$-algebra with Fr-action. We can also define the graded version $\check{S}_A^\bullet$, where we put $V_A^\vee$ in degree 2.

remark 5.2.2.1. Notice that the Tate module $T_\ell(A)$ is equivalently the pro-ℓ part of $\pi_1^{\mathrm{tame}}(A)$, since $A[\ell^k]$ can be identified as the Galois group of the ℓ^k-th power covering $A \to A$.

Corollary 5.2.2.21. Take any torus A, we have an equivalence of categories

$$D_m(\mathbb{B}A) \simeq D\left(\check{S}_A, \mathrm{Fr}\right)^\omega$$

The pullback functor $D_m(\mathbb{B}A) \to D_m(\mathrm{pt}) = D(\mathrm{Fr})$ corresponds to the functor $(-) \otimes_{\check{S}_A} \overline{\mathbb{Q}}_\ell$ using the natural augmentation of $\check{S}_A$.

Proof: By Theorem 5.2.2.7, it suffices to verify that $D_m(\mathbb{B}A)$ satisfies the assumptions Assumption 5.2.2.6. For (1), we simply pick the trivial decomposition. For (2), we take $\mathcal{C}_\lambda$ to be the very pure of weight 0 complexes (see Definition 5.2.3.7) such that they are constant over $\overline{k} = \overline{\mathbb{F}}_q$. Then, the $\mathrm{Ext}^i_{D_\lambda}(c_1, c_2)$ would be pure of weight i by the same argument as in Proposition 5.2.3.10, whose Frobenius unipotent part obviously vanish. They are clearly stable under taking tensors with unipotent Frobenius modules.

For (3), we take $\mathbb{1}_\lambda$ to be the constant sheaf. Since the torus A is connected, the space $\mathbb{B}A$ is simply connected, so $D_m(\mathbb{B}A)$ is generated by the object $\mathbb{1}_\lambda$, let along the subcategory $\mathcal{C}$ of very pure of weight 0 complexes. By definition, every constant very pure of weight 0 complex is the tensor product of $\mathbb{1}_\lambda$ with some complex of Frobenius module M. The condition (4) is obvious since there is only 1 stratum.

Finally, the essential image of the fully faithful embedding is given by Corollary 5.2.2.19, which is $D\left(\check{S}_A, \mathrm{Fr}\right)^\omega$, or equivalently, the smallest triangulated subcategory generated by the constant sheaf.

The second assertion follows from the functoriality of $\check{S}_A$ by Theorem 5.2.2.7. □

5.2.3. Application 1: The equivariant (irreducible) side

In this section, we will consider the properties of the category $D_m(B\backslash G/B)$ and its parabolic variant $D_m(P_\Theta\backslash G/B)$.

Definition 5.2.3.1. For each $\overline{w} \in [W_\Theta\backslash W]$, we define the following objects in $D_m(P_\Theta\backslash G/B)$:

$$\Delta_{\overline{w}} = i_{\overline{w},!}\overline{\mathbb{Q}}_\ell[\ell(\overline{w})](\ell(\overline{w})/2), \quad \nabla_{\overline{w}} = i_{\overline{w},*}\overline{\mathbb{Q}}_\ell[\ell(\overline{w})](\ell(\overline{w})/2), \quad \mathrm{IC}_{\overline{w}} = i_{\overline{w},!*}\overline{\mathbb{Q}}_\ell[\ell(\overline{w})](\ell(\overline{w})/2)$$

The projection ${}_\Theta\pi : \mathrm{Fl} \to {}_\Theta\mathrm{Fl}$ is proper, so we obtain two pairs of adjunctions

$$\begin{tikzcd}
D_m(P_\Theta\backslash G/B) \arrow[rr, "{}_\Theta\pi_*"] \arrow[rr, phantom, shift left=4, "\perp"] \arrow[rr, phantom, shift right=4, "\perp"] & & D_m(B\backslash G/B) \arrow[ll, bend right=50, "{}_\Theta\pi^*"'] \arrow[ll, bend left=50, "{}_\Theta\pi^!"]
\end{tikzcd}$$

We will first investigate the relations between the categories $D_m(P_\Theta\backslash G/B)$ through the projection ${}_\Theta\pi$.

Lemma 5.2.3.2. For $w \in W$, we write $w = u\overline{w}$ for $u \in W_\Theta$ and $\overline{w} \in [W_\Theta\backslash W]$. Then, we have

$${}_\Theta\pi_*(\Delta_w) = \Delta_{\overline{w}}[-\ell(u)](-\ell(u)/2), {}_\Theta\pi_*(\nabla_w) = \nabla_{\overline{w}}[\ell(u)](\ell(u)/2)$$

Proof: Firstly, notice that the map of cells $\mathrm{Fl}_w \to {}_\Theta\mathrm{Fl}_{\overline{w}}$ is the projection

$${}_\Theta\pi_w : \mathbb{A}^{\ell(w)} = \mathbb{A}^{\ell(u)} \times \mathbb{A}^{\ell(\overline{w})} \twoheadrightarrow \mathbb{A}^{\ell(\overline{w})}$$

Then, we have the computation

$$\begin{aligned}
{}_\Theta\pi_*(\Delta_w) = {}_\Theta\pi_!(\Delta_w) &= {}_\Theta\pi_! i_{w,!}\overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2) \\
&= i_{\overline{w},!}\,{}_\Theta\pi_{w,!}\overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2) \\
&= i_{\overline{w},!}\overline{\mathbb{Q}}_\ell[\ell(w)-2\ell(u)]((\ell(w)-2\ell(u))/2) \\
&= \left(i_{\overline{w},!}\overline{\mathbb{Q}}_\ell[\ell(\overline{w})](\ell(\overline{w})/2)\right)[-\ell(u)](-\ell(u)/2) \\
&= \Delta_{\overline{w}}[-\ell(u)](-\ell(u)/2)
\end{aligned}$$

The pushforward of ∇_w is computed likewise. □

The computation of pushforward of IC_w is much more harder. We postpone this to the later part of this section (Lemma 5.2.3.18). In fact, we can not give a complete characterization of this pushforward, but only a few things on its components.

Lemma 5.2.3.3. We have a natural isomorphism of functors ${}_\Theta\pi^*{}_\Theta\pi_*(-) \simeq \mathbb{1}_\Theta *_B (-)$. Here, $\mathbb{1}_\Theta$ is the constant sheaf on $\mathrm{Fl}_{\leqslant w_\Theta}$ (w_Θ is the maximal length element in W_Θ), so is equivalently $\mathrm{IC}_{w_\Theta}[-\ell_\Theta](-\ell_\Theta/2)$.

Proof: Firstly note that we have the following Cartesian diagram

$$\begin{tikzcd}
(B_\Theta\backslash L_\Theta)\times^{B_\Theta}(U_\Theta\backslash G) \arrow[r, "m"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\ulcorner", very near start] & B\backslash G \arrow[d, "{}_\Theta\pi"] \\
B\backslash G \arrow[r, "{}_\Theta\pi"'] & P_\Theta\backslash G
\end{tikzcd}$$

This square is Cartesian because we can explicitly construct a map

$$(B_\Theta\backslash L_\Theta)\times^{B_\Theta}(U_\Theta\backslash G) \to (B\backslash G)\times_{P_\Theta\backslash G}(B\backslash G), (B_\Theta l, U_\Theta g) \mapsto (Blg, Bg)$$

and this is an isomorphism by the decomposition $B = U_\Theta B_\Theta$, and hence we have an inverse map $(Bg_1, Bg_2) \mapsto (B_\Theta l, U_\Theta g_2)$, where l is the unique element such that $Bg_1 = Blg_2$.

By the proper base change along the Cartesian diagram, we have ${}_\Theta\pi^*{}_\Theta\pi_*(-) \simeq m_* p_2^*(-)$. Then, in the coordinates $(B_\Theta\backslash L_\Theta)\times^{B_\Theta}(U_\Theta\backslash G)$, we can write $p_2^*(\mathcal{F}) = \mathbb{1}_\Theta \boxtimes^{B_\Theta} \mathcal{F}$, where $B_\Theta\backslash L_\Theta$ is identified as ${}_\Theta\pi^{-1}(P_\Theta\backslash P_\Theta) \subset B\backslash G$. Finally, $m_*(\mathbb{1}_\Theta \boxtimes^{B_\Theta} \mathcal{F}) = \mathbb{1}_\Theta *_B \mathcal{F}$, because

$$(B_\Theta\backslash L_\Theta)\times^{B_\Theta}(U_\Theta\backslash G) = (B_\Theta\backslash L_\Theta)\times^B G = {}_\Theta\pi^{-1}(P_\Theta\backslash P_\Theta)\times^B G \subset \mathrm{Fl}\times^B G$$

□

Definition 5.2.3.4. Besides the tensor product, there is a convolution product on $D_m(B\backslash G/B)$, given by the diagram

$$B\backslash G/B \times B\backslash G/B \xleftarrow{\delta} B\backslash G\times^B G/B \xrightarrow{m} B\backslash G/B$$

where δ sends (g_1, g_2) to (g_1, g_2) is well-defined, since the middle quotient splits into 2 quotients. We then define $\mathcal{F}_1 *_B \mathcal{F}_2 = m_!\delta^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$. Note that here m is ind-proper and δ is a quotient by B hence smooth.

Lemma 5.2.3.5. If $u, v \in W$ is such that $\ell(u) + \ell(v) = \ell(uv)$, then

$$\Delta_u *_B \Delta_v = \Delta_{uv}, \quad \nabla_u *_B \nabla_v = \nabla_{uv}$$

Proof: Follows from that $\mathrm{Fl}_u \times^B \mathrm{Fl}_v \to \mathrm{Fl}_{uv}$ is an isomorphism. $\square$

Proposition 5.2.3.6. For $\mathcal{F}_1, \mathcal{F}_2 \in D_m(B\backslash G/B)$ very pure of weight 0, we have $\mathcal{F}_1 *_B \mathcal{F}_2$ very pure of weight 0.

Proof: Note that the category of very pure complexes of weight 0 is

$$\langle \Delta_w | w \in W\rangle \cap \langle \nabla_w | w \in W\rangle$$

where $\langle - \rangle$ denote the additive category generated by some objects under extensions. Then, we have

$$\Delta_u *_B \Delta_v \in \langle \Delta_w\langle \leqslant 0\rangle | w \in W\rangle$$

and

$$\nabla_u *_B \nabla_v \in \langle \nabla_w\langle \geqslant 0\rangle | w \in W\rangle$$

by writing the standard and costandard sheaves into convolutions of standard and costandard sheaves associated to simple reflections using Lemma 5.2.3.5. The convolution $\Delta_s *_B \Delta_s$ is a very classical computation using excision on $\mathbb{P}^1$, and we invite the readers not familiar with the result to compute it by themselves.

Thus, we have

$$\mathcal{F}_1 *_B \mathcal{F}_2 \in \langle \Delta_w\langle \leqslant 0\rangle | w \in W\rangle \cap \langle \nabla_w\langle \geqslant 0\rangle | w \in W\rangle$$

which implies $\mathcal{F}_1 *_B \mathcal{F}_2$ is pure of weight 0. Hence by Lemma 5.2.3.9, it is very pure of weight 0. $\square$

Definition 5.2.3.7. Let $X = \coprod X_\lambda$ be a stratified scheme, and suppose $\mathcal{F} \in D_m(X)$ is constructible with respect to the stratification. Then, let $i_\lambda : X_\lambda \hookrightarrow X$ denote the embeddings, we say $\mathcal{F}$ is $*$-pure or !-pure of weight n if for each λ and each $m \in \mathbb{Z}$, we have $H^m(i_\lambda^*(\mathcal{F}))$ or $H^m\big(i_\lambda^!(\mathcal{F})\big)$ is a sheaf pointwise pure of weight $m + n$. We say $\mathcal{F}$ is very pure of weight 0 if it is both !-pure and $*$-pure of weight 0.

Lemma 5.2.3.8. Let V be a variety over k, equipped with a $\mathbb{G}_m$-action contracting V to a point $v \in V$. Then, we have for any complex of $\mathbb{G}_m$-equivariant constructible ℓ-adic sheaf $\mathcal{F}$, the cohomology $H^*(V, \mathcal{F}) = \mathcal{F}_v$.

Proof: By the spectral sequence computing the hypercohomology, it suffices to prove that for any $\mathbb{G}_m$-equivariant ℓ-adic sheaf $\mathcal{F}$, the cohomology $H^*(V, \mathcal{F}) = \mathcal{F}_v$ (in particular, the higher cohomologies vanishes). Then, by replacing $\mathcal{F} = \ker(\mathcal{F} \to v_* v^* \mathcal{F})$, we may assume $\mathcal{F}_v = 0$. We now prove $\mathcal{F}$ has trivial cohomology.

We denote $\mu : \mathbb{G}_m \times V \to V$, and extend μ naturally to $\mu : \mathbb{A}^1 \times V \to V$, where $(0, u) \mapsto v$ for any $u \in V$. Then, consider the shearing map $\tau : \mathbb{A}^1 \times V \to \mathbb{A}^1 \times V$, which sends $(t, u) \mapsto (t, \mu(t, u))$. Note that τ is an $\mathbb{A}^1$-morphism, where $\mathbb{A}^1 \times V$ is equipped with the $\mathbb{A}^1$-scheme structure by the map $\pi_1 : \mathbb{A}^1 \times V \to \mathbb{A}^1$.

Since $\mathcal{F}$ is $\mathbb{G}_m$-equivariant, we have $\pi_2^*(\mathcal{F})|_{\mathbb{G}_m \times V} = \tau^*\pi_2^*(\mathcal{F})|_{\mathbb{G}_m \times V}$. Then, $\tau^*\pi_2^*(\mathcal{F})|_{0 \times V} = 0$ by assumption, so if we let $j : \mathbb{G}_m \times V \hookrightarrow \mathbb{A}^1 \times V$ denote the inclusion, we have

$$\tau^*\pi_2^*(\mathcal{F}) = i_! i^* \pi_2^*(\mathcal{F})$$

Applying the counit $i_! i^* \to \mathrm{id}$ and the derived projection to $\mathbb{A}^1$, we obtain a map

$$\alpha : \pi_{1,*}\tau^*\pi_2^*(\mathcal{F}) \to \pi_{1,*}\pi_2^*(\mathcal{F})$$

On the other hand, since we have the commutative triangle

$$\begin{tikzcd} \mathbb{A}^1 \times V \arrow[rr, "\tau"] \arrow[dr, "\pi_1"'] & & \mathbb{A}^1 \times V \arrow[dl, "\pi_1"] \\ & V & \end{tikzcd}$$

we have a natural transformation $\pi_{1,*} \to \pi_{1,*}\tau^*$ given by $\pi_{1,*} = \pi_{1,*}\tau_*$, so

$$\pi_{1,*} \xrightarrow{\eta} \pi_{1,*}\tau_*\tau^* \xrightarrow{\simeq} \pi_{1,*}\tau^*$$

Applying this to $\pi_2^*(\mathcal{F})$, we obtain a map

$$\beta : \pi_{1,*}\pi_2^*(\mathcal{F}) \to \pi_{1,*}\tau^*\pi_2^*(\mathcal{F})$$

Notice that $\pi_{1,*}\pi_2^*(\mathcal{F})$ is the constant sheaf with values $\pi_{2,*}(\mathcal{F})$ by smooth base change. Also, note the stalk of $\alpha \circ \beta$ at $1 \in \mathbb{A}^1$ is bijective, and the stalk of this map at 0 is 0. Since it is a constant sheaf, it must be 0 everywhere. $\square$

Lemma 5.2.3.9. Suppose $\mathcal{F} \in D_m(P_\Theta\backslash G/B)$ is pure of weight 0. Then it is very pure of weight 0.

Proof: Firstly, we may assume $\Theta = \varnothing$, because the projection ${}_\Theta\pi$ is smooth and ${}_\Theta\pi^*$ preserves weights. By definition, we have for any $x \in B\backslash G/B$, the $*$-stalk $x^*(\mathcal{F})$ having weight $\leqslant 0$. So it suffices to prove that the $*$-stalks have weights $\geqslant 0$. For this, we think $\mathcal{F}$ as an equivariant sheaf over Fl, and first assume $\operatorname{Supp} \mathcal{F} \subset \mathrm{Fl}_{\leqslant w}$. Consider $v \leqslant w$, and we abuse notations to let $v = vB/B \in \mathrm{Fl}$. Then, it suffices to prove $v^*(\mathcal{F})$ has weight $\geqslant 0$ for all v, since they are representatives of all the orbits. But this is because

$$v^*(\mathcal{F}) = H^*(vC \cap \mathrm{Fl}_{\leqslant w}, \mathcal{F})$$

by Lemma 5.2.1.12 and Lemma 5.2.3.8. Then, since H^* is the $*$-pushforward to the point, it increases weight, so $v^*(\mathcal{F})$ has weight $\geqslant 0$. Hence the assertion follows. $\square$

Proposition 5.2.3.10. If $\mathcal{F}_1 \in D_m(B\backslash G/B)$ is $*$-pure of weight 0, and $\mathcal{F}_2 \in D_m(B\backslash G/B)$ is !-pure of weight 0, then $\mathrm{Ext}^i(\mathcal{F}_1, \mathcal{F}_2)$ is a Frobenius module of weight i, and it is free over the left or right copy of the graded algebra $\check{S}^\bullet$. Here, the $\check{S}^\bullet$ bimodule structure is given by the action

$$D_m(H\backslash\mathrm{pt}/H) \circlearrowleft D_m(B\backslash G/B)$$

Proof: We apply induction on the support of $\mathcal{F}_1$ and $\mathcal{F}_2$. Suppose the statement is true for $\mathcal{F}_1$ and $\mathcal{F}_2$ supported on $\mathrm{Fl}_{<w}$. Then, suppose now they are supported on $\mathrm{Fl}_{\leqslant w}$, consider the excision triangle

$$i_{<w,!}i^!_{<w}(\mathcal{F}_2) \to \mathcal{F}_2 \to i_{w,*}i^*_w(\mathcal{F}_2)$$

and applying $\mathrm{Hom}(\mathcal{F}_1, -)$, we obtain a long exact sequence

$$\cdots \to \mathrm{Ext}^i\left(i^*_{<w}(\mathcal{F}_1), i^!_{<w}(\mathcal{F}_2)\right) \to \mathrm{Ext}^i(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Ext}^i(i^*_w(\mathcal{F}_1), i^*_w(\mathcal{F}_2)) \to \cdots$$

Now, note i_w is open, and by assumption $i^*_w(\mathcal{F}_1)$ and $i^*_w(\mathcal{F}_2) = i^!_w(\mathcal{F}_2)$ are both pure of weight 0. We now prove that we have $\mathrm{Ext}^i(i^*_w(\mathcal{F}_1), i^*_w(\mathcal{F}_2))$ pure of weight i. Firstly, we write $\mathcal{G}_i = i^*_w(\mathcal{F}_i)$. Then, since $\mathcal{G}_i \in D_m(B\backslash BwB/B) = D_m(B\backslash \mathrm{Fl}_w)$, we have the stabilizer of the actino $B \circlearrowleft \mathrm{Fl}_w$ given by $B_w = B \cap wBw^{-1}$, which is constant. Thus, the underlying sheaf of $\mathcal{G}_i$ in $D_m(\mathrm{Fl}_w)$ is constant. We may assume $\mathcal{G}_i = \mathcal{L}_w \otimes M_i$ non-canonically, where $\mathcal{L}_w = \overline{\mathbb{Q}}_\ell[d](d/2)$ is the constant sheaf on the strata and M_i is a Frobenius module pure of weight 0. Moreover, we may assume M_i is dualizable. Hence, we have the following computation

$$\mathrm{Hom}^\bullet(\mathcal{G}_1, \mathcal{G}_2) = \mathrm{Hom}^\bullet(\mathcal{L}_w, \mathcal{L}_w \otimes M_2 \otimes M_1^\vee) = H^\bullet(B\backslash \mathrm{Fl}_w) \otimes \mathrm{Hom}^\bullet(M_1, M_2)$$

Now, by definition, $\mathrm{Ext}^i(M_1, M_2)$ is pure of weight i, and since $B\backslash \mathrm{Fl}_w$ is equivalent to $\mathbb{B}H$ for the torus $H \subset G$, $H^i(B\backslash \mathrm{Fl}_w) = H^i(\mathbb{B}H)$ is also pure of weight i. Hence, taking tensor products gives the desired purity result.

On the other hand, $\mathrm{Ext}^i\left(i^*_{<w}(\mathcal{F}_1), i^!_{<w}(\mathcal{F}_2)\right)$ is pure of weight i by induction hypothesis. Then, since we have an actual map of Frobenius modules

$$\mathrm{Ext}^i(i^*_w(\mathcal{F}_1), i^*_w(\mathcal{F}_2)) \to \mathrm{Ext}^{i+1}\left(i^*_{<w}(\mathcal{F}_1), i^!_{<w}(\mathcal{F}_2)\right)$$

by weight reasons, it must be 0. Hence, the long exact sequence splits into short exact sequences

$$0 \to \mathrm{Ext}^i\left(i^*_{<w}(\mathcal{F}_1), i^!_{<w}(\mathcal{F}_2)\right) \to \mathrm{Ext}^i(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Ext}^i(i^*_w(\mathcal{F}_1), i^*_w(\mathcal{F}_2)) \to 0$$

so $\mathrm{Ext}^i(\mathcal{F}_1, \mathcal{F}_2)$ is pure of weight i. Then, it remains to prove $\mathrm{Ext}^\bullet(i^*_w(\mathcal{F}_1), i^*_w(\mathcal{F}_2))$ is free both as a left $\check{S}^\bullet$-module and as a right $\check{S}^\bullet$-module. This is also by the computation above, since by definition $H^\bullet(B\backslash \mathrm{Fl}_w) = H^\bullet(\mathbb{B}H)$. □

Then we will introduce the hypercohomology functor for the convenience for certain computations.

Definition 5.2.3.11. By identifying $B\backslash G/B = H\backslash(U\backslash G/U)/H$, we can project $U\backslash G/U$ to a point, and obtain the equivariant cohomology functor

$$\Gamma_{H\times H}(U\backslash G/U, -) : D_m(B\backslash G/B) \to D_m(\mathbb{B}(H \times H)) = D\left(\check{S} \otimes \check{S}, \mathrm{Fr}\right)^\omega$$

where $\check{S} = \check{S}_H$. We denote this functor by $\mathbb{H}$.

Lemma 5.2.3.12. The functor $\mathbb{H}$ has a natural monoidal structure with respect to the convolution on $D_m(B\backslash G/B)$ and the tensor product $- \otimes_{\check{S}} -$ on $D\left(\check{S} \otimes \check{S}, \mathrm{Fr}\right)^\omega$.

Proof: It suffices to prove that the tensor product $- \otimes_{\check{S}} -$ on $D\left(\check{S} \times \check{S}, \mathrm{Fr}\right)^{\omega}$ is the same as the following pull-push:

$$H\backslash\mathrm{pt}/H \times H\backslash\mathrm{pt}/H \xleftarrow{\delta_H} H\backslash\mathrm{pt} \times^H \mathrm{pt}/H \xrightarrow{m_H} H\backslash\mathrm{pt}/H$$

then, clearly the convolutions (here we use $m_* \delta^*$, which makes no difference on $D_m(B\backslash G/B)$ by ind-properness) on the corresponding diagrams commutes with the usual pushforward functor $\pi_* : D_m(B\backslash G/B) \to D_m(H\backslash\mathrm{pt}/H)$ (essentially by smooth base change and ignoring trivial unipotent actions).

On the level of derived categories, we have the following identifications:

$$\begin{tikzcd}
D_m(H\backslash\mathrm{pt}/H \times H\backslash\mathrm{pt}/H) \arrow[r, "\delta_H^*"] \arrow[d, "\simeq"'] & D_m(H\backslash\mathrm{pt} \times^H \mathrm{pt}/H) \arrow[r, "m_{H,*}"] \arrow[d, "\simeq"'] & D_m(H\backslash\mathrm{pt}/H) \arrow[d, "\simeq"] \\
D\left(\check{S}^{\otimes 4}, \mathrm{Fr}\right) \arrow[r, "F"] & D\left(\check{S}^{\otimes 3}, \mathrm{Fr}\right) \arrow[r, "U"] & D\left(\check{S} \otimes \check{S}, \mathrm{Fr}\right)
\end{tikzcd}$$

Here, by Corollary 5.2.2.21 F is $- \otimes_{\check{S} \otimes \check{S}} \check{S}$, where the bottom $\check{S} \otimes \check{S}$ is the two in the middle. The question now is what is U. We claim that this is forgetting the middle $\check{S}$-action, hence identifying $U \circ F$ as the tensor product.

We first identify the functor $\pi_* : D_m(\mathbb{B}A) \to D_m(\mathrm{pt})$. While identifying $D_m(\mathbb{B}A)$ with $D\left(\check{S}_A, \mathrm{Fr}\right)$, we used the functor $\mathrm{Hom}(\mathbb{1}, -)$, which is precisely taking the cohomology. Thus π_* is simply the forgetful functor $D\left(\check{S}_A, \mathrm{Fr}\right) \to D(\mathrm{Fr})$. Hence in the above case, this is also the forgetful functor, but only forgetting the middle factor by the Künneth formula. $\square$

Lemma 5.2.3.13. For $\mathcal{F} \in D_m\left(B\backslash\overline{BwB}/B\right)$ $*$-pure or !-pure of weight 0, we have exact sequences

$$0 \longrightarrow \mathbb{H}\left(j_! j^! \mathcal{F}\right) \longrightarrow \mathbb{H}(\mathcal{F}) \longrightarrow \mathbb{H}(i_* i^*(\mathcal{F})) \longrightarrow 0$$

$$0 \longrightarrow \mathbb{H}\left(i_! i^! \mathcal{F}\right) \longrightarrow \mathbb{H}(\mathcal{F}) \longrightarrow \mathbb{H}(j_* j^*(\mathcal{F})) \longrightarrow 0$$

here $i : B\backslash\mathrm{Fl}_{<w} \hookrightarrow B\backslash\mathrm{Fl}_{\leqslant w}$, and $j : B\backslash\mathrm{Fl}_w \hookrightarrow B\backslash\mathrm{Fl}_{\leqslant w}$ are the closed and open part. We temporarily denote them by $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$.

Proof: It suffices to prove the statements for the second exact sequence, since the others follows from taking Verdier duality. For $\mathcal{F}$!-pure of weight 0, this case follows from Proposition 5.2.3.10 and weight reasons.

Now, for $\mathcal{F}$ $*$-pure of weight 0, suppose $\mathcal{F} = i_{\leqslant v}^* \mathcal{K}$. By the long exact sequence, it suffices to show that $j^* : H^\bullet(X, \mathcal{F}) \to H^\bullet(U, \mathcal{F})$ is degreewise surjective. Let v_H denote the inclusion of stacks $v_H : H\backslash v \hookrightarrow H\backslash\mathrm{Fl}$. By $\mathbb{A}^1$-invariance, we have an isomorphism of cohomology groups

$$H^\bullet(B\backslash\mathrm{Fl}_G, \mathcal{K}) \simeq H^\bullet(H\backslash\mathrm{Fl}_G, \mathcal{K})$$

Then, one has a commutative diagram

$$\begin{tikzcd}
H^\bullet(B\backslash \mathrm{Fl}, \mathcal{K}) \arrow[r] \arrow[d, equal] & H^\bullet(X, \mathcal{F}) \arrow[r, "j^*"] & H^\bullet(U, \mathcal{F}) \arrow[r, equal] & H^\bullet v_H^* \mathcal{F} \arrow[d, equal] \\
H^\bullet(H\backslash \mathrm{Fl}, \mathcal{K}) \arrow[rr, "\alpha^*"] & & H^\bullet(H\backslash vC, \mathcal{K}) \arrow[r, equal] & H^\bullet v_H^* \mathcal{F}
\end{tikzcd}$$

Here, the rightmost horizontal arrows are taking stalks at the contraction point, so by Lemma 5.2.3.8 is an isomorphism. Now, to prove j^* surjective, it suffices to prove α^* surjective, where $\alpha : H\backslash vC \hookrightarrow H\backslash \mathrm{Fl}$ is an open immersion of stacks. We take β to be the reduced complement closed of α, we have

$$\cdots \longrightarrow H^k(H\backslash \mathrm{Fl}, \mathcal{K}) \longrightarrow H^k(H\backslash vC, \mathcal{K}) \longrightarrow H^{k+1}\big(H\backslash(\mathrm{Fl} - vC), \beta^! \mathcal{K}\big) \longrightarrow \cdots$$

Note that we may assume $\mathcal{K}$ is pure of weight 0. Hence, $\beta^!(\mathcal{K})$ is of weight $\geqslant 0$. Thus, the Fr-module $H^{k+1}\big(H\backslash(\mathrm{Fl} - vC), \beta^! \mathcal{K}\big)$ is of weight $\geqslant k+1$. On the other hand, as a Fr-module we have $H^k(H\backslash vC, \mathcal{K}) = H^k(v_H^*(\mathcal{K}))$, so this group has weight k. Hence the connecting morphism is 0, so we obtain the surjectivity of α^*. $\square$

The main evidence that $\mathbb{H}$ is useful for detecting data in $D_m(B\backslash G/B)$ is given by the following proposition.

Proposition 5.2.3.14. For $\mathcal{F}_1, \mathcal{F}_2 \in D_m(B\backslash G/B)$ very pure of weight 0, the natural map

$$\mathrm{Ext}^\bullet(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{Hom}_{\check{S}\otimes\check{S}}(\mathbb{H}(\mathcal{F}_1), \mathbb{H}(\mathcal{F}_2))$$

is an isomorphism of Fr-modules.

Proof: We first prove that for $\mathcal{F}_1, \mathcal{F}_2 \in D_m(B\backslash G/B)$ supported on $B\backslash\overline{BwB}/B$ very pure of weight 0, we have $\mathrm{Hom}^\bullet(\mathcal{F}_1, \mathcal{F}_2) \simeq \mathrm{Hom}^\bullet_{A_{\leqslant w}}(\mathbb{H}(\mathcal{F}_1), \mathbb{H}(\mathcal{F}_2))$. Here $A_{\leqslant w} := H^\bullet(B\backslash \mathrm{Fl}_{\leqslant w})$. We apply induction on the Bruhat order, and prove that for any v, we have

$$\mathrm{Hom}^\bullet\big(i_{\leqslant v,*} i^*_{\leqslant v}(\mathcal{F}_1), i_{\leqslant v,*} i^!_{\leqslant v}(\mathcal{F}_2)\big) \simeq \mathrm{Hom}_{A_{\leqslant v}}\big(\mathbb{H}\big(i_{\leqslant v,*} i^*_{\leqslant v}(\mathcal{F}_1)\big), \mathbb{H}\big(i_{\leqslant v,*} i^!_{\leqslant v}(\mathcal{F}_2)\big)\big)$$

The assertion for $v = e$ is trivial. Then, for the induction step, we should apply Lemma 5.2.3.13. We claim that we have a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}^\bullet\big(i_* i^*(\mathcal{F}_1), i_* i^!(\mathcal{F}_2)\big) \arrow[r] \arrow[d] & \mathrm{Hom}_{A_{<v}}\big(\mathbb{H}(i_* i^*(\mathcal{F}_1)), \mathbb{H}\big(i_* i^!(\mathcal{F}_2)\big)\big) \arrow[d] \\
\mathrm{Hom}^\bullet(\mathcal{F}_1, \mathcal{F}_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{A_{\leqslant v}}(\mathbb{H}(\mathcal{F}_1), \mathbb{H}(\mathcal{F}_2)) \arrow[d] \\
\mathrm{Hom}^\bullet(j_* j^*(\mathcal{F}_1), j_* j^*(\mathcal{F}_2)) \arrow[r] \arrow[d] & \mathrm{Hom}_{H^\bullet(U)}(\mathbb{H}(j_* j^*(\mathcal{F}_1)), \mathbb{H}(j_* j^*(\mathcal{F}_2))) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Now, we know that the left line is an exact sequence, the right line is exact at top and bottom, and that the top and bottom horizontal arrows are isomorphisms, either by the induction hypothesis or by the computation on the case of a torus (note $U = B\backslash BwB/B \simeq H\backslash\mathrm{pt}$ up to factors of $\mathbb{A}^1$'s). Now, it suffices to prove that the right line is exact in the middle, so we can finish the induction step.

Firstly, note that we have an exact sequence

$$0 \longrightarrow H_c^\bullet(U) \longrightarrow A_{\leqslant v} \longrightarrow A_{<v} \longrightarrow 0$$

By the above equivalence, $H^\bullet(U)$ is a free $\check{S}$-module of rank 1, so we may choose a generator $[U] \in H_c^{2\ell(v)}(U)$. Then, the action of $[U]$ on $\mathbb{H}(\mathcal{F}_i)$ would then kill the closed part, hence factors as

$$\mathbb{H}(\mathcal{F}_i) \twoheadrightarrow \mathbb{H}(j_*j^*(\mathcal{F}_i)) \xrightarrow{u} \mathbb{H}(j_!j^*(\mathcal{F}_i))[2\ell(v)](\ell(v)) \hookrightarrow \mathbb{H}(\mathcal{F}_i)[2\ell(v)](\ell(v))$$

Now, suppose there is some $\varphi : \mathbb{H}(\mathcal{F}_1) \to \mathbb{H}(\mathcal{F})_2$ induces the zero map $\mathbb{H}(j_*j^*(\mathcal{F}_1)) \to \mathbb{H}(j_*j^*(\mathcal{F}_2))$, then we have $[U] \circ \varphi = 0$ by the above factoring. Hence, $\varphi \circ [U] = 0$ since φ is an $A_{\leqslant v}$-module map. Thus $\varphi = 0$ on the image of $[U]$, which is $\mathbb{H}(j_!j^*(\mathcal{F}_1))$. Thus it comes from an $A_{<v}$-module map $\mathbb{H}(i_*i^*(\mathcal{F}_1)) \to \mathbb{H}\big(i_*i^!(\mathcal{F}_2)\big)$, hence the claim follows. □

Then, we can compute the equivariant cohomology of the standard, costandard, and intersection cohomology sheaves.

Lemma 5.2.3.15. For each $w \in W$, we have an isomorphism in $\mathbf{Mod}^{\mathrm{gr}}_{\check{S}^\bullet\otimes\check{S}^\bullet,\,\mathrm{Fr}}$

$$\mathbb{H}(\Delta_w) = \mathcal{O}_{\Gamma(w)}[-\ell(w)](-\ell(w)/2)$$

$$\mathbb{H}(\nabla_w) = \mathcal{O}_{\Gamma(w)}[\ell(w)](\ell(w)/2)$$

Here, $\Gamma(w) \subset V_H \times V_H = \operatorname{Spec} \check{S} \otimes \check{S}$ is the graph of the map $w : V_H \to V_H$, given by the action $W \circlearrowright V_H$.

Proof: Note that the projection $B\backslash G/B \to \mathrm{pt}$ is ind-proper, so the hypercohomologies in question are actually given by

$$\mathbb{H}(\Delta_w)[-\ell(w)](-\ell(w)/2) = H_c^\bullet\big(B\backslash BwB/B, \overline{\mathbb{Q}}_\ell\big), \quad \mathbb{H}(\nabla_w)[\ell(w)](\ell(w)/2) = H^\bullet\big(B\backslash BwB/B, \overline{\mathbb{Q}}_\ell\big)$$

We first compute the cohomology of the costandard object. Notice that by Lemma 5.2.3.8 we have an isomorphism $H^\bullet\big(B\backslash BwB/B, \overline{\mathbb{Q}}_\ell\big) = H^\bullet\big(H\backslash HwH/H, \overline{\mathbb{Q}}_\ell\big)$. We then suffices to identify it with the ring of functions over $\mathcal{O}_{\Gamma(w)}$. Note the $H \times H$-action on HwH is given by $(h_1, h_2) \cdot x = h_1 x h_2^{-1}$, so the stabilizer is given by $K_w = \big\{(whw^{-1}, h) | h \in H\big\}$. Thus, we can furthermore identify the stack $H\backslash HwH/H = \mathbb{B}K_w$. However, by the identifying K_w with H, we have $H \simeq K_w \hookrightarrow H \times H$ induces $V_H \hookrightarrow V_H \times V_H$, $v \mapsto (wv, v)$ on the level of Tate modules, hence its image is precisely $\Gamma(w)$. Hence the claim for the second part follows.

For the first part, note that

$$H_c^\bullet\big(B\backslash BwB/B, \overline{\mathbb{Q}}_\ell\big) = H^\bullet\big(H\backslash HwH/H, \iota^!\overline{\mathbb{Q}}_\ell\big)$$

where $\iota : H\backslash HwH/H \hookrightarrow B\backslash BwB/B$ is the inclusion of the left H-contraction point, and by Lemma 5.2.3.8, we have $H^\bullet(-) = \iota^*(-)$, so by taking Verdier dual $H_c^\bullet(-) = \iota^!(-)$. This time, by purity for the section of an affine bundle, we have $\iota^!\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell[-2\ell(w)](-\ell(w))$. Hence the conclusion follows. □

remark 5.2.3.1. The cohomologies are actually isomorphism to each other if we forget the Tate twists, cohomological grading, and $X_*(H)$-gradings. Indeed, we have $\mathcal{O}_{\Gamma(w)} = \mathcal{O}_{V_H}$ regarded as a plain $\check{S}$-module.

The cohomologies of IC_w's are much more interesting. They can be described as the algebra of functions over a union of the orbits, which gives a algebro-geometric description of Soergel bimodules. For example, it is well-known that in the case $G = \mathrm{SL}_2$, $\mathbb{H}(\mathrm{IC}_s)$ is the Soergel bimodule $\check{S} \otimes_{\check{S}^s} \check{S}$. This can be interpreted by definition as $\mathcal{O}(\Gamma(e) \cup \Gamma(s))[1](1/2)$.

Lemma 5.2.3.16. For $w \in W$, we have $\mathbb{H}_w := \mathbb{H}(\mathrm{IC}_w)$ is a direct sum of Fr-modules $\overline{\mathbb{Q}}_\ell[n](n/2)$ for $n \equiv \ell(w) (\mathrm{mod}\, 2)$.

Proof: For the case where w is a simple reflection, it follows from the computation of SL_2. In general, take a reduced word presentation $w = s_{i_1} \cdot \cdots \cdot s_{i_m}$, and we claim

$$\mathrm{IC}_w \subset \mathrm{IC}_{s_{i_1}} *_B \cdots *_B \mathrm{IC}_{s_{i_n}}$$

is a direct summand. Note

$$\mathrm{End}\Big(\mathrm{IC}_{s_{i_1}} *_B \cdots *_B \mathrm{IC}_{s_{i_n}}\Big) \subset \mathrm{End}_{\check{S} \otimes \check{S}}\Big(\mathbb{H}_{s_{i_1}} \otimes_{\check{S}} \cdots \otimes_{\check{S}} \mathbb{H}_{s_{i_n}}\Big)$$

is the 0-th graded direct summand and the latter is Frobenius semisimple since it is an endomorphism Fr-module of a semisimple Fr-module, and by Corollary 5.2.2.17 $\mathrm{IC}_w \otimes M_w$ would be a direct summand. However M_w must be $\overline{\mathbb{Q}}_\ell$ by counting the dimension of $H^{\ell(w)}(-)$. So IC_w is indeed a direct summand.

Hence $\mathbb{H}_w$ is a direct summand of $\mathbb{H}_{s_{i_1}} \otimes_{\check{S}} \cdots \otimes_{\check{S}} \mathbb{H}_{s_{i_n}}$, and in the latter, only $\overline{\mathbb{Q}}_\ell[n](n/2)$ appears for $n \equiv \ell(w) (\mathrm{mod}\, 2)$ □

remark 5.2.3.2. I believe the proof using Bott-Samuelson resolutions is also available.

Lemma 5.2.3.17. For $w_1, w_2 \in W$, the convolution $\mathrm{IC}_{w_1} *_B \mathrm{IC}_{w_2}$ is a direct sum of $\mathrm{IC}_w[n](n/2)$'s where $n \equiv \ell(w) - \ell(w_1) - \ell(w_2) (\mathrm{mod}\, 2)$. In particular, if $\ell(w)_1 + \ell(w_2) = \ell(2)$, then IC_w appears with multiplicity 1.

Proof: Let $\mathcal{F} = \mathrm{IC}_{w_1} *_B \mathrm{IC}_{w_2}$. Again, we have

$$\mathrm{End}(\mathcal{F}) \subset \mathrm{End}_{\check{S} \otimes \check{S}}\Big(\mathbb{H}_{w_1} \otimes_{\check{S}} \mathbb{H}_{w_2}\Big)$$

is the 0-th graded direct summand. Hence $\mathcal{F}$ decomposes into $\mathrm{IC}_w \otimes M_w$ for Fr-semisimple modules. Then, we have $\mathbb{H}(\mathcal{F}) = \mathbb{H}_{w_1} \otimes_{\check{S}} \mathbb{H}_{w_2}$ is a direct sum of $\mathbb{H}_w \otimes M_w$. Hence by Lemma 5.2.3.16, we can compute M_w and see that it is a direct sum of $\overline{\mathbb{Q}}_\ell[n](n/2)$ with the desired parity.

In the case $\ell(w_1) + \ell(w_2) = \ell(w)$, the map $\overline{Bw_1B} \times^B \overline{Bw_2B} \to \overline{BwB}$ is birational, hence the multiplicity 1 follows. □

Lemma 5.2.3.18. For $w \in W$, the complex ${}_\Theta\pi_*(\mathrm{IC}_w)$ is a direct sum of $\mathrm{IC}_{\overline{v}}[n](n/2)$ with $n \equiv \ell(\overline{w}) - \ell(\overline{v}) (\mathrm{mod}\, 2)$. In particular, if $w \in [W_\Theta \backslash W]$, we have $\mathrm{IC}_{\overline{w}}$ a direct summand of ${}_\Theta\pi_*(\mathrm{IC}_w)$ with multiplicity 1.

Proof: Again we compute the endomorphism algebra

$$\mathrm{End}_{D_m(P_\Theta \backslash G/B)}({}_\Theta\pi_*(\mathrm{IC}_w)) \simeq \mathrm{Hom}_{D_m(B\backslash G/B)}(\mathbb{1}_\Theta *_B \mathrm{IC}_w, \mathrm{IC}_w)$$

and the latter is again Frobenius semisimple by applying $\mathbb{H}$. Hence ${}_\Theta\pi_*(\mathrm{IC}_w)$ is a direct sum of $\mathrm{IC}_{\overline{v}} \otimes M_{\overline{v}}$, hence ${}_\Theta\pi^* {}_\Theta\pi_*(\mathrm{IC}_w) = \mathbb{1}_\Theta *_B \mathrm{IC}_w$ is a direct sum of ${}_\Theta\pi^*(\mathrm{IC}_{\overline{v}}) \otimes M_{\overline{v}}$. Applying $\mathbb{H}$ and again use Lemma 5.2.3.16 we have the statement for the parity. The multiplicity 1 again follows from birationality. □

Finally, we show that we can apply Theorem 5.2.2.7 to give a computation of $D_m(B\backslash G/B)$, but we will postpone the actual computation to later parts.

Proposition 5.2.3.19. Suppose X is any scheme with an action of a reductive group G having finitely many orbits X_λ. Moreover, assume each orbit inclusion $X_\lambda \hookrightarrow X$ is affine. Then, the category $D_m(G\backslash X)$ satisfies Assumption 5.2.2.6.

Proof:
(1) The finite poset is given by Λ, with the order given by taking boundaries. Each strata is given by $\mathcal{D}_\lambda = D_m(G\backslash X_\lambda)$.
(2) For each $D_m(G\backslash X_\lambda)$, we take $\mathcal{C}_\lambda$ to be the subcategory of constant pure of weight 0 complexes. Thus by a similar argument as Proposition 5.2.3.10, the Frobenius unipotent part of the higher Ext groups vanish.
(3) We can take $\mathbb{1}_\lambda$ to be the constant sheaf, and since $\mathcal{C}_\lambda$ consists of constant complexes, every $c \in \mathcal{C}_\lambda$ is of the form $\mathbb{1}_\lambda \otimes M$. The generation is because $G\backslash X_\lambda$ is of the form $\mathbb{B}G_\lambda$ where G_λ is the stabilizer of some point. Then, G_λ is connected so $\pi_1(\mathbb{B}G_\lambda) = 0$. Hence $D_m(\mathbb{B}G_\lambda)$ is generated by the constant sheaf.
(4) We take $c_\lambda = \mathrm{IC}_\lambda$. By irreducibility, we have the map $\mathrm{End}(c_\lambda) = \mathrm{End}(\mathbb{1}_\lambda)$ is an isomorphism.

□

Corollary 5.2.3.20. $D_m\left(P_\Theta \backslash \overline{P_\Theta \overline{w} B}/B\right)$ satisfies Assumption 5.2.2.6.

5.2.4. Basics on (completed) monodromic categories

We first give a recap on the definition of monodromic sheaves. In this part, all monodromic means unipotently monodromic.

Definition 5.2.4.1. Recall $\widetilde{\mathrm{Fl}} = G/U$ is the enhanced flag variety. We define $D_m^{\mathrm{mon}}(U\backslash G/U)$ to be the subcategory of $D_m(B\backslash G/B)$ spanned by the image of $\pi^! : D_m(U\backslash G/U) \to D_m(B\backslash G/B)$. Equivalently, we may take $D_m^{\mathrm{mon}}(U\backslash G/U)$ to be the subcategory of $D_m(B\backslash G/B)$ spanned by the image of $\pi^! : D_m(U\backslash G/B) \to D_m(B\backslash G/B)$ by stratawise computations.

However, this category is not what we need. Rather, we need a certain completion of this category. We can see this from the following example.

Example 5.2.4.1. Take $G = \mathbb{G}_m$. Then, the category $D_m^{\mathrm{mon}}(\mathbb{G}_m)$ is the subcategory spanned by the constant sheaves. We let $\mathrm{Loc}_u(\mathbb{G}_m)$ denote the Abelian category of unipotent local systems over $\mathbb{G}_m$ (i.e., local systems $\mathcal{L}$ over $\mathbb{G}_m$ equipped with a filtration $0 = \mathcal{L}_0 \subset \cdots \subset \mathcal{L}_n = \mathcal{L}$ such that $\mathcal{L}_i/\mathcal{L}_{i-1}$ is constant). Then, we claim that we have the following characterization:

$$D_m^{\mathrm{mon}}(\mathbb{G}_m) = \{\mathcal{F} \in D_m(\mathbb{G}_m), \mathcal{H}^i(\mathcal{F}) \in \mathrm{Loc}_u(\mathbb{G}_m)\}$$

Indeed, the right hand side is closed under taking kernels, cokernels, extensions, and direct summands, and contains the constant sheaf. Conversely, every unipotent local system must lie in $D_m^{\mathrm{mon}}(\mathbb{G}_m)$ by considering the filtration, so the two categories are equivalent.

Then, we can identify $\mathrm{Loc}(\mathbb{G}_m)$ with ℓ-adic representations of $\pi_1(\mathbb{G}_m)$. Such representations then factors through its pro-ℓ part $T_\ell(\mathbb{G}_m) = \mathbb{Z}_\ell(1)$, so we have $\mathrm{Loc}(\mathbb{G}_m) = \mathbf{Rep}\left(\mathbb{Z}_\ell(1), \overline{\mathbb{Q}}_\ell^n\right)$. The unipotent local systems hence corresponds to unipotent representations.

Now, for unipotent representations over a field of characteristic 0, we should pass to the nilpotent monodromy operator. Namely, for a representation $\rho : T_\ell(\mathbb{G}_m) \to \mathrm{GL}\left(\overline{\mathbb{Q}}_\ell^n\right)$, we have a linear map

$$\log \rho : V_{\mathbb{G}_m} \to \mathrm{End}\left(\overline{\mathbb{Q}}_\ell^n\right),\ v \mapsto \log \rho(v), v \in T_\ell(\mathbb{G}_m)$$

If we choose a basis $v_0 \in V$ (V is 1-dimensional), clearly the data of $\log \rho$ is completely determined by $N := \log \rho(v_0)$. Hence, $\mathrm{Loc}_u(\mathbb{G}_m) = \mathbf{Mod}^{f.l.}_{\overline{\mathbb{Q}}_\ell[[t]],\mathrm{Fr}}$, where $\mathrm{Fr}(t) = q^{-1}t$, and $f.l.$ denote finite length. Thus, we have by Morita theory $D_m^{\mathrm{mon}}(\mathbb{G}_m) = D_{f.l.}\left(\overline{\mathbb{Q}}_\ell[[t]], \mathrm{Fr}\right)$.

Notice that the whole category $D\left(\overline{\mathbb{Q}}_\ell[[t]], \mathrm{Fr}\right)$ is a completion of $D_{f.l.}\left(\overline{\mathbb{Q}}_\ell[[t]], \mathrm{Fr}\right)$. So we see that we need a completion $\hat{D}_m^{\mathrm{mon}}(\mathbb{G}_m)$ instead. In particular, we will need the object corresponding to the module $\overline{\mathbb{Q}}_\ell[[t]]$.

Construction 5.2.4.2. Recall that $\mathrm{Pro}(D_m^{\mathrm{mon}}(U\backslash G/U))$ admits a natural stable ∞-structure.

5.2.5. Application 2: The monodromic (tilting) side

6. The Bezrukavnikov-Arkhipov equivalence

See my thesis.

7. Mixed hodge modules

7.1. Pure and mixed hodge structures

7.1.1. Preliminaries on filtered objects

Definition 7.1.1.1. We fix a category $\mathcal{C}$. We define the category of filtered to be the category of functors $F : \mathbb{Z} \to \mathcal{C}$ such that each morphism is injective. A filtration on an object $c \in \mathcal{C}$ is a functor $\mathbb{Z} \to \mathrm{Sub}_c$, the category of subobjects of c.

We define a morphism of filtered vector spaces to be $f : c \to d$, such that $f(F_i c) \subset F_i d$. We say the morphism is strictly compatible with the filtration if we have an exact sequence

$$0 \longrightarrow \operatorname{gr} \ker f \longrightarrow \operatorname{gr} V \longrightarrow \operatorname{gr} W \longrightarrow \operatorname{gr} \operatorname{coker} f \longrightarrow 0$$

Equivalently, we must have $f(F_n c) = f(c) \cap F_n d$.

We define a decreasing filtered object to be $\mathbb{Z}^{\mathrm{op}} \to \mathcal{C}$. We denote decreasing filtrations by $F^\bullet c$, while increasing filtrations by $F_\bullet c$. The translation between increasing filtered objects and decresing filtered objects is given by the convention $F^n c = F_{-n} c$.

Construction 7.1.1.2. Let $F_\bullet c \in \mathbf{Alg}(\mathrm{Fil}(\mathcal{C}))$, then we define the Rees construction of c to be the graded ring

$$\tilde{c} = \bigoplus_{p \in \mathbb{Z}} F_p c \lambda^p$$

which is a graded $\mathbb{1}_{\mathcal{C}}[\lambda^{\pm 1}]$-algebra.

Warning 7.1.1.3. For an Abelian category $\mathcal{A}$, the category of filtered objects $\mathrm{Fil}(\mathcal{A})$ is generally not Abelian. This can be seen in the example in $\mathbf{FilVect}_{\mathbb{C}}$: take $F^\bullet V$ with $F^i V = \mathbb{C}$ for $i < 0$ and 0 for $i \geqslant 0$, and $F^\bullet W$ with $F^i W = \mathbb{C}$ for $i \leqslant 0$ and 0 for $i > 0$. Then, $f : F^\bullet V \to F^\bullet W$ corresponding to id on the underlying space has trivial kernel and cokernel, but is not an isomorphism.

However, the above example is the only part where Abelianess fails. If we take the wide subcategory $\mathrm{Fil}(\mathcal{A})^{\mathrm{str}}$ with morphisms strictly compatible with filtrations, we get an Abelian category. In fact by [Sta26] 05SI, we have a morphism is strict iff its coimage coincide with the image, and other axioms for Abelian categories holds already for filtered objects by [Sta26] 0122.

7.1.2. Basic definitions

Definition 7.1.2.1. We define a pure hodge structure of weight w to be a finite dimensional $\mathbb{C}$-vector space $\mathcal{H}$ equipped with two decreasing filtrations $F'^\bullet\mathcal{H}$ and $F''^\bullet\mathcal{H}$, such that

$$F'^p\mathcal{H} \oplus F''^{w-p+1}\mathcal{H} = \mathcal{H}$$

Equivalently, $\mathcal{H}$ admit a bigraded direct sum decomposition

$$\mathcal{H} = \bigoplus_{p \in \mathbb{Z}} \mathcal{H}^{p, w-p}$$

The two definitions can be translated to the other version as follows:

$$\mathcal{H}^{p,w-p} = F'^p\mathcal{H} \cap F''^{w-p}\mathcal{H}, \qquad F'^p\mathcal{H} = \bigoplus_{q\leqslant p} \mathcal{H}^{p,w-p}, F''^{w-p}\mathcal{H} = \bigoplus_{q\geqslant p} \mathcal{H}^{p,w-p}$$

We define a morphism of Hodge structures to be a linear map between the underlying vector spaces compatible with both filtrations. We denote the category of Hodge structures pure of some weight to be $\mathrm{HS}(\mathbb{C})$, and the category of Hodge structures pure of weight w to be $\mathrm{HS}(\mathbb{C}, w)$.

Construction 7.1.2.2. $\mathrm{HS}(\mathbb{C})$ admits a closed monoidal structure. For $\mathcal{H}_1$ of weight w_1 and $\mathcal{H}_2$ of weight w_2, we may define $\mathcal{H}_1 \otimes \mathcal{H}_2$ to be the pure Hodge structure of weight $w_1 + w_2$, with

$$(\mathcal{H}_1 \otimes \mathcal{H}_2)^{k,w_1+w_2-k} = \bigoplus_p \left(\mathcal{H}_1^{p,w_1-p} \otimes \mathcal{H}_2^{k-p,w_2-k+p}\right)$$

Then, for $\mathcal{H}$ pure of weight w, we may define the dual $\mathcal{H}^\vee$ pure of weight $-w$ by

$$(\mathcal{H}^\vee)^{-k,-w+k} = \left(\mathcal{H}^{k,w-k}\right)^\vee$$

Thus, we have $\mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{H}_1^\vee \otimes \mathcal{H}_2$, which is pure of weight $w_2 - w_1$.

Then, we define the complex conjugate of $\mathcal{H}$ to be a Hodge structure of the same weight, with $\overline{\mathcal{H}}^{k,w-k} = \overline{\mathcal{H}^{k,w-k}}$. For the Hermitian dual, we define $\mathcal{H}^* = \overline{\mathcal{H}^\vee} = \overline{\mathcal{H}}^\vee$.

Proposition 7.1.2.3. $\mathrm{HS}(\mathbb{C}, w)$ is a semisimple Abelian category. $\mathrm{HS}(\mathbb{C})$ is not Abelian.

Proof: We prove a morphism in $\mathrm{HS}(\mathbb{C}, w)$, compatible with both filtrations, is strictly compatible with both filtrations. Notice first that $f : \mathcal{H}_1 \to \mathcal{H}_2$ preserves the direct sum decomposition, namely $f\left(\mathcal{H}_1^{p,w-p}\right) \subset \mathcal{H}_2^{p,w-p}$. This is equivalent to the strict compatiblity. □

Hence, the above caveat shows that $\mathrm{HS}(\mathbb{C})$ is not the correct category to consider. Instead, we must pass to the Rees construction to give the correct definition.

8. Nori motives

9. Gestalten

9.1. Higher presentable categories

9.1.1. Presentability

We first recall the following definitions of presentable categories. As usual, we fix a regular cardinal κ.

Definition 9.1.1.1. Let $\mathcal{C}$ be an $(\infty,1)$-category and $c \in \mathcal{C}$ an object. We say a diagram in $\mathcal{C}$ is κ-filtered if any subdiagram of size $< \kappa$ can be extended to a right cone.

We say c is κ-compact if $\mathrm{Hom}(c,-)$ commutes with κ-filtered colimits. We denote the κ-compact objects by $\mathcal{C}^\kappa$.

We say $\mathcal{C}$ is κ-presentable if there is a subset $S \subset \mathcal{C}$ of κ-compact objects such that $\mathcal{C}$ is generated by these κ-compact objects under colimits. By Theorem 5.5.1.1 of [Lur08], this is equivalent to the condition $\mathcal{C} = \mathrm{Ind}_\kappa(\mathcal{C}^\kappa)$.

We say $\mathcal{C}$ is presentable if it is κ-presentable for some κ. Again by Theorem 5.5.1.1 of [Lur08], this is equivalent to the condition that $\mathcal{C}$ is a localization of a presheaf category.

Definition 9.1.1.2. We define $\mathbf{Pr}^L$ to be the non-locally small $(\infty,1)$-category whose objects are the class of the presentable categories and morphisms the colimit preserving functors (technically speaking, we cannot define $\mathbf{Pr}^L$ because its morphisms form a class). We define $\mathbf{Pr}^L_\kappa \subset \mathbf{Pr}^L$ to be the 1-full subcategory with objects the κ-presentable categories and morphisms the functors preserving κ-compact objects. This time it is indeed locally small.

Theorem 9.1.1.3. We have $\mathbf{Pr}^L_\kappa \in \mathbf{Pr}^L_\kappa$.

Proof: By the equivalent incarnation of κ-presentable categories, we have $\mathbf{Pr}^L_\kappa$ equivalent to the categories that are of the form $\mathcal{C}^\kappa$ for $\mathcal{C} \in \mathbf{Pr}^L_\kappa$. This condition is formulated as $\mathcal{C}^\kappa$ is small, admits all κ-small colimits and retracts, and the functors between them preserving κ-small colimits. We denote this equivalent category by Rex_κ. We now prove Rex_κ is κ-presentable.

Note that the forgetful functor $U : \mathrm{Rex}_\kappa \to \mathbf{Cat}$ admit a left adjoint given by $\mathcal{C} \mapsto \mathrm{pSh}(\mathcal{C})^\kappa$. We claim that this is a pair of monadic adjunction, so that Rex_κ is the algebra of the monad $U \circ F$. By Proposition 4.7.3.5 of [Lur17], it suffices to prove that U is conservative, Rex_κ admits and U preserves U-split geometric realizations. Clearly U is conservative (it is faithful). Then, let $\mathcal{C}_\bullet$ be a simplicial diagram in Rex_κ, admitting a geometric realization $\mathcal{C}_{-1} \in \mathbf{Cat}$, with a splitting of $\mathcal{C}_\bullet \to \mathcal{C}_{-1} \in \mathbf{Cat}$. But then, $\mathrm{Fun}(I,\mathcal{C}_\bullet) \to \mathrm{Fun}(I,\mathcal{C}_{-1})$ is a split geometric realization in $\mathbf{Cat}$ for any κ-small diagram I, extending to $I^\triangleright$ shows that $\mathcal{C}_{-1} \in \mathrm{Rex}_\kappa$.

Finally, we use this to show that Rex_κ is κ-presentable. Firstly it admit all colimits because it is the algebra of a monad, and $\mathbf{Cat}$ admits all colimits. Note that $\mathbf{Cat}$ is the category of all small categories, hence compactly generated by the categories Δ^n. Upon showing U preserves κ-filtered colimits, we can show F preserves κ-compact objects, which shows that Rex_κ is κ-compactly generated. For a κ-filtered diagram $\mathcal{D}_i \in \mathrm{Rex}_\kappa$, their κ-filtered colimit in $\mathbf{Cat}$ $\mathcal{D}$ admits all κ-small colimits by the construction of colimit categories, and the functors $\mathcal{D}_i \to \mathcal{D}$ preserves κ-small colimits. This finishes the proof. □

We recall more facts on the category $\mathbf{Pr}^L$.

Theorem 9.1.1.4 (Adjoint functor theorem, [Lur08] Corollary 5.5.2.9). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable categories. Then
(1) F admits a right adjoint iff F preserves all colimits.
(2) F admits a left adjoint iff F preserves all limits and preserves κ-filtered colimits for some regular cardinal κ. In this case, its left adjoint preserves κ-compact objects.

Definition 9.1.1.5. We define $\mathbf{Pr}^R$ and $\mathbf{Pr}^R_\kappa$ to be the categories of (κ-) presentable categories with morphisms right adjoints. They are opposite to $\mathbf{Pr}^L$ and $\mathbf{Pr}^L_\kappa$ respectively.

Theorem 9.1.1.6 ([Lur17], Proposition 5.5.3.13, Theorem 5.5.3.18, Proposition 5.5.7.6). The category $\mathbf{Pr}^L$ admits all limits, and the forgetful functor $\mathbf{Pr}^L \to \widehat{\mathbf{Cat}}$ preserves limits.

The category $\mathbf{Pr}^L$ admits all colimits, equivalently, the category of $\mathbf{Pr}^R$ admits all limits, and the forgetful functor $\mathbf{Pr}^R \to \widehat{\mathbf{Cat}}$ preserves limits.

The analogous statements hold for $\mathbf{Pr}^L_\kappa$ and $\mathbf{Pr}^R_\kappa$.

Then, we recall the definition of Lurie tensor products of presentable categories.

Construction 9.1.1.7. For $\mathcal{C}, \mathcal{D} \in \mathbf{Pr}^L$, there is a category $\mathcal{C} \otimes \mathcal{D} \in \mathbf{Pr}^L$, universal among presentable categories with a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$. An analogous construction works for $\mathcal{C}, \mathcal{D} \in \mathbf{Pr}^L_\kappa$.

The construction goes as follows: suppose $\mathcal{C} = \mathrm{pSh}(\mathcal{C}_0)[W^{-1}]$, and $\mathcal{D} = \mathrm{pSh}(\mathcal{D}_0)[V^{-1}]$, we have

$$\mathcal{C} \otimes \mathcal{D} = \mathrm{pSh}(\mathcal{C}_0 \times \mathcal{D}_0)[W^{-1} \times V^{-1}]$$

It can be verified that this gives a symmetric monoidal structure on $\mathbf{Pr}^L$ and $\mathbf{Pr}^L_\kappa$.

Theorem 9.1.1.8. The category $\mathbf{Pr}^L$ is Cartesian closed. The internal Hom object is given by $\mathrm{Fun}^L(-,-)$. Similarly, the category of $\mathbf{Pr}^L_\kappa$ has internal Hom object $\mathrm{Ind}_\kappa \mathrm{Fun}^{\mathrm{Rex}_\kappa}((-)^\kappa, (-)^\kappa)$.

10. Appendix

10.1. Filtered objects

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